THE TUNNELING EFFECT AND SOME IMPLICATIONS

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Abstract: It is shown that the tunneling effect can, mathematically, be described by the tunneling operator. Since the tunneling operator is invertible, it is possible to specify the form of the wave inside the barrier and from it arrive at the incident wave in the nontunneling region. An easy example of this is presented. On the other hand, this operator allows us to build a no-moving localized structure inside a tunneling barrier at any distance, as far as one wish, independent of time. This seems to imply an instantaneous transit time inside the barrier or, in other words, an infinite velocity. The overall process is here applied to a photonic localized structure in the so-called classical frustrated total reflection where all space besides the crystal can be taken as the tunneling barrier. The process, based on wavelet analysis, avoids the problems raised by Fourier nonlocal and nontemporal paradigm. It overcomes the difficult, for not saying impossible, problem of the definition of the "true" velocity of a wave, other than the one of the harmonic plane wave.

Key words: fundamental quantum physics, tunneling, tunneling operator, superluminal velocities, optics, wavelet local analysis.

1. INTRODUCTION

The tunneling effect has been studied for many years but until now no one, as far as I know, has ever tried to find an operator to describe it mathematically. The advantage of an operator for describing this effect results from the fact that it allows us to pass from the knowledge of the form of the wave in one region to expression of the wave in the other region. A most simple example of it is shown. By stipulating the analytic form of the wave in the tunneling region, we directly arrive at the wave in the "normal" region. Furthermore, the tunneling operator allows us to study in a very easy and intuitive way the complex problem of the superluminal transit times.

Recently many experiments [1] have been done showing, without any margin of doubts, that the pulses, which cross the tunneling barrier, arrive before the ones going through the air. Everybody agrees with the results of the experiments. The question is if they imply a superluminal velocity or not? Since all calculations are done under Fourier nonlocal and nontemporal paradigm, [2] is very difficult, for not saying impossible, to define the velocity of a wave. Only in the case of an infinite, both in time and in space, harmonic plane wave the problem offers no difficulty. In this nonlocal paradigm, only the infinite harmonic plane wave has a "true" velocity and frequency. All other finite waves are no more than combinations of these infinite waves. Therefore, any of these finite waves has, in principle, as many velocities and frequencies as the number of harmonic plane waves that make it. Since these harmonic plane waves are spread over the whole space and time, his hard to tell, in this paradigm, which is the velocity of a finite wave [3]. Since the components that make the pulse fill all space and time, practically anything is possible. The weirdest things are then possible like, for instance, retroactions in time. That is, a pulse may arrive at a destination prior being produced by the source! Situations of this kind are not new in physics. We have just to recall the description and prediction, in geophysics, of earthquakes. The use of Fourier analysis implies, in certain cases, that the seismic pulses arrive before the actual earthquake took place. Other times the precursors started before the formation of the Earth itself! Since geophysicists are people facing everyday practical concrete problems, they need to have their feet placed on solid ground, so they discard, as a nuisance, these aberrant situations resulting from the use of Fourier global paradigm. It was precisely to avoid these irregular situations, that the geophysicist, Jean Morlet, created, in the eighties, the concept of finite wave from which wavelet local analysis sprout [4].

Here we follow a more general approach[2], inspired in the local analysis, for the tunneling process. This method allows us to overcome the above conceptual and practical difficulties raised by the usual nonlocal and nontemporal Fourier paradigm which, in this case, is overcome by a local paradigm.

2. THE TUNNELIN OPERATOR

The tunneling effect means the passage from a "normal" region of space to other characterized by a potential with energy greater than the energy of the incident particle. This action also can, mathematically, be described by the following operation

$$f_N(kx-\omega t) \xrightarrow{Tunneling} f_T(ik'x-\omega t) = T\ f_N(kx-\omega t)\,, \qquad (1)$$

with T standing for the tunneling operator, and f being a generic solution of the wave equation.

This means that the passage from a "normal" region to a tunneling one is to be described by changing, in the solution of the wave equation, the classical velocity by an imaginary velocity, or equivalently: k by ik'. The passage from a tunneling region to a "normal" one is done under the same tunneling operator

$$f_T(ik'x-\omega t) \xrightarrow{Tunneling} f_N(k''x-\omega t) = T\ f_T(ik'x-\omega t)\,. \qquad (1')$$

It will be shown that, in certain conditions, the time does not appear in the mathematical expression describing the no-moving localized structure. This means that the no-moving structure can be localized at any point inside the barrier independently of time. Accepting this conclusion by its straight value, forgetting that the mathematical formulas we use are only wrought, better or worse, approximations for describing Reality, it would imply an instantaneous time transmission.

3. TUNNELING WAVES WITH A CONSTANT INTENSITY

Before showing, with a concrete example, that the tunneling operator has indeed the above properties we shall apply it to a most simple example. The result is, as we shall see, in a certain sense, the one to be expected. By applying the tunneling operator to the wave, solution to the classical wave equation, of the form

$$e^{(kx-\omega t)} \xrightarrow{T} e^{(ik'x-\omega t)}, \qquad (2)$$

we, obtain a wave, which intensity inside the tunneling region, is constant all along the *xx* axis

$$I_T = \psi_T \psi_T^* = C\, e^{(ik'x-\omega t)} C^*\, e^{(-ik'x-\omega t)} = |C|^2\, e^{-2\omega t} \qquad (3)$$

The incident wave in the normal region

$$\psi_{NI} = f(kx-\omega t) = A e^{(kx-\omega t)} \qquad (4)$$

is a crescent wave from $-\infty \xrightarrow{to} 0$, with a maximum at point zero, and decreasing in time.

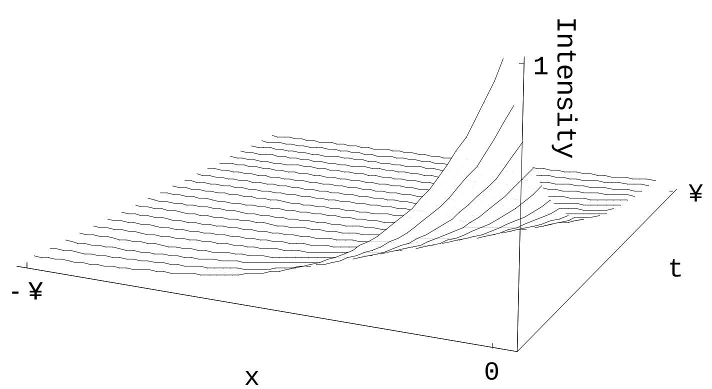

Fig.1 – Plot of the incident wave

For the reflected wave, the case is symmetrical, since the function starts decreasing from the zero origin to minus infinity, $0 \xrightarrow{to} -\infty$. Summing up, everything happens as if the boundary between the two regions behaved like a mirror giving origin to a virtual reflected wave, as indicated in Fig.2.

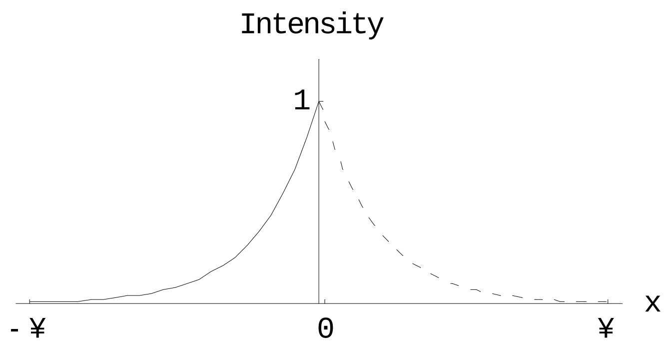

Fig.2 – Incident and virtual reflected wave

Since we are dealing, in practice, with a decreasing function its derivative must be negative. In this condition, taking in account the above considerations, the total wave in the normal region is to be written

$$\psi_N = \left(A e^{kx} + B e^{-k|x|}\right) e^{-\omega t}, \tag{5}$$

and in the tunneling zone

$$\psi_T = C e^{ikx} e^{-\omega t}. \tag{6}$$

By imposing the condition of continuity of the function in all domain, we have at the boundary *x=0*

$$\psi_N(0,t) = \psi_T(0,t), \quad \text{and} \quad \psi'_N(0,t) = \psi'_T(0,t), \tag{7}$$

giving

$$\begin{cases} (A+B)e^{-\omega t} = C e^{-\omega t} \\ k(A-B)e^{-\omega t} = ik' C e^{-\omega t} \end{cases} \quad \text{or} \quad \begin{cases} (A+B) = C \\ k(A-B) = ik' C \end{cases} \tag{8}$$

that is

$$B = \frac{k - ik'}{k + k'} A \tag{9}$$

with $|B|^2 = |A|^2$, and

$$C = \frac{2k}{k + ik'} A . \qquad (10)$$

The plot of the intensities of the incident and tunnel wave intensity, at a very short time near zero, is shown in Fig.3

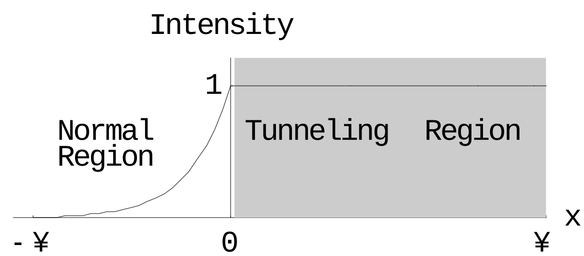

Fig.3 – Plot of the incident and tunnel wave intensity, at a very short time near zero

This result, as we have said, ought to be expected. Since a wave of constant amplitude, a plane wave, incident on an infinite tunneling barrier gives rise, inside it, to an evanescent wave the inverse operation is also true. An evanescent wave in the normal region gives origin to a wave of constant intensity (3) in the tunneling region.

4. THE TUNNELING OPERATOR *T* IN THE CASE OF CLASSICAL TOTAL FRUSTRATED REFLECTION

These calculations can be found in any good textbook of optics [5], nevertheless for the sake of clarity some steps will be shown. Consider Fig.4 showing two optical mediums with refractive indexes n_1 and n_2, such that $n_1>n_2$, and a beam of light with an incident angle θ_1. Furthermore, for simplicity reasons, we assume that the second medium is the air, therefore we have $n_2 \approx 1$.

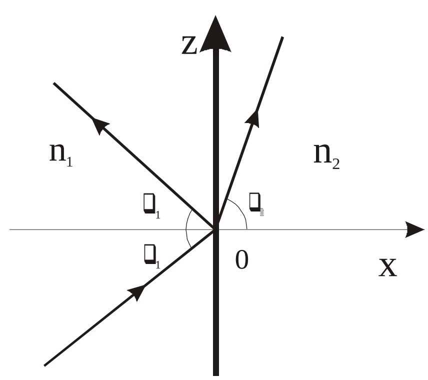

Fig.4 – Refraction of light

In the region 2 the wave vector can be written

$$\vec{k}_2 = k_2 \cos\theta_2\, \vec{e}_x + k_2 \sin\theta_2\, \vec{e}_z \,. \qquad (11')$$

The problem now is to express this function in terms of the angle of incidence θ_1. From Snell refraction law and taking in consideration that we are interested in the tunneling situation, it is necessary to impose the total reflection condition. Therefore, after some trivial calculations we arrive at

$$\vec{k}_2 = i k_2 \left(\tfrac{k_1^2}{k_2^2} \sin^2\theta_1 - 1\right)^{\frac{1}{2}} \vec{e}_x + k_1 \sin\theta_1\, \vec{e}_z \,. \qquad (11)$$

This expression indicates that the component of the vector $\vec{k}_2$ along the penetration direction *xx* has the generic form

$$i\,k' = i\,k_2 \left(\tfrac{k_1^2}{k_2^2} \sin^2\theta_1 - 1\right)^{\frac{1}{2}} \qquad (12)$$

with

$$k' = k_2 \left(\tfrac{k_1^2}{k_2^2} \sin^2\theta_1 - 1\right)^{\frac{1}{2}} . \qquad (13)$$

This clearly shows that the passage from a "normal" region to other under tunneling conditions is indeed described by the tunneling operator T such that

$$k \quad \xrightarrow{T=Tunneling} \quad i\,k' \,. \qquad (14)$$

The same conclusion could also have been be reached if instead of a classical calculation we had done a quantum one for the case of penetration of a potential barrier [6] when the energy of the incident particle is less than the energy of the barrier.

5. A NO-MOVING LOCALIZED STRUCTURE INSIDE A TUNNELING BARRIER

Probably there must happen to be many processes for obtaining a no-moving localized structure inside a tunneling barrier. Nevertheless, we shall refer here only the one that uses a wave with amplitude of the type

$$e^{(k(x-\ell)-\omega t)^2} \quad \xrightarrow{T} \quad e^{(i k'(x-\ell)-\omega t)^2} = e^{-k'^2 (x-\ell)^2} e^{\omega^2 t^2} e^{2 i k (x-\ell)\omega t} \,. \qquad (15)$$

In the case under consideration, see Fig.4, we have for the wave vector

$$\begin{cases} \vec{k}_1 = k_1 \cos\theta_1\, \vec{e}_x + k_1 \sin\theta_1\, \vec{e}_z \\ \vec{k}_2 = i\,k_2 \left(\tfrac{k_1^2}{k_2^2} \sin^2\theta_1 - 1\right)^{\frac{1}{2}} \vec{e}_x + k_2 \sin\theta_2\, \vec{e}_z \end{cases} \qquad (16)$$

or remembering that

$$\begin{cases} k = k_1 \cos\theta_1 \\ k' = k_2 (\frac{k_1^2}{k_2^2} \sin^2 \theta_1 - 1)^{\frac{1}{2}} \end{cases} \tag{17}$$

and Snell law, the wave vector can be written

$$\begin{cases} \vec{k}_1 = k\,\vec{e}_x + k_1 \sin\theta_1\,\vec{e}_z \\ \vec{k}_2 = i\,k'\vec{e}_x + k_1 \sin\theta_1\,\vec{e}_z \end{cases} \tag{18}$$

under such conditions the wave, solution of the wave equation, in the two regions has the form

$$\begin{cases} \psi_1(x,z,t) = \left[A e^{\frac{(kx-\omega t+\varepsilon)^2}{\alpha^2} + i\,kx} + B e^{\frac{(-kx-\omega t+\varepsilon)^2}{\alpha^2} - i\,kx} \right] e^{i(k_1 \sin\theta_1 z - \omega t)} \\ \psi_2(x,z,t) = C\,e^{\frac{(i\,k'x-\omega t+\varepsilon)^2}{\alpha^2} - k'x}\, e^{i(k_1 \sin\theta_1 z - \omega t)} \end{cases} \tag{19}$$

These expressions indicate that the *zz* components are equal. This result should be expected since along that direction there is no discontinuity. The expressions (19) can also be written

$$\begin{cases} \psi_1(x,z,t) = \psi_1(x,t)\,e^{i(k_1 \sin\theta_1 z - \omega t)} \\ \psi_2(x,z,t) = \psi_2(x,t)\,e^{i(k_1 \sin\theta_1 z - \omega t)} \end{cases} \tag{19'}$$

In order to fix the values of the constants it is necessary to impose the customary continuity boundary conditions

$$\begin{cases} \psi_1(0,t) = \psi_2(0,t) \\ \psi'_1(0,t) = \psi'_2(0,t) \end{cases} \tag{20}$$

These calculations leading to

$$\begin{aligned} A + B &= C \\ k(A - B) &= ik'C \end{aligned} \tag{21}$$

which give

$$B = \frac{k - ik'}{k + ik'}\,A \tag{22}$$

and

$$C = \frac{2k}{k + ik'}\,A\,, \tag{23}$$

with

$$|B|^2 = |A|^2 . \tag{24}$$

In order to introduce a localized no-moving structure inside the barrier it is necessary to find the right form for the parameter ε. In such conditions, it is convenient to have for the amplitude of the incident wave

$$e^{\frac{(k(x-\ell)-\omega t+\varepsilon_i)^2}{\alpha^2}} = e^{\frac{(kx-\omega t+\varepsilon)^2}{\alpha^2}} \tag{25}$$

this shows that we need to have

$$\varepsilon = -k\ell + \varepsilon_i . \tag{26}$$

The tunneling operator applied to (25) gives

$$T\left\{ e^{\frac{(k(x-\ell)-\omega t+\varepsilon_i)^2}{\alpha^2}} \right\} = e^{\frac{(ik'(x-\ell)-\omega t+\varepsilon_t)^2}{\alpha^2}} = e^{\frac{(ik'x-\omega t+\varepsilon)^2}{\alpha^2}} , \tag{27}$$

with

$$\varepsilon = -ik' + \varepsilon_t . \tag{28}$$

Developing (27) we got

$$e^{\frac{(ik'(x-\ell)-\omega t+\varepsilon_t)^2}{\alpha^2}} = e^{-\frac{(x-\ell)^2}{\alpha^2/k'^2}} \; e^{\frac{(-\omega t+\varepsilon_t)^2}{\alpha^2}} \; e^{2ik'(x-\ell)(-\omega t+\varepsilon_t)} . \tag{29}$$

The value of ε_t must be such as to preserve the localized structure

$$e^{-\frac{(x-\ell)^2}{\alpha^2/k'^2}} . \tag{30}$$

From (19) it can be seen that for the amplitude of the reflected wave we have

$$e^{\frac{(-kx-\omega t+\varepsilon)^2}{\alpha^2}} = e^{\frac{(k(x-\ell)+\omega t+\varepsilon_r)^2}{\alpha^2}} , \tag{31}$$

with

$$\varepsilon = k\ell - \varepsilon_r . \tag{32}$$

Writing the three equalities (26), (32) and (28)

$$\begin{cases} \varepsilon = -k\ell + \varepsilon_i \\ \varepsilon = +k\ell - \varepsilon_r \\ \varepsilon = -ik'\ell + \varepsilon_t \end{cases}$$

which shows that

$$\varepsilon_r = 2k\ell - \varepsilon_i\,, \tag{33}$$

and

$$\varepsilon_t = -(k - ik')\ell + \varepsilon_i\,. \tag{34}$$

It is not convenient to make $\varepsilon_r = \varepsilon_i$, because it would imply $\varepsilon = 0$, thus conducing to the loss of the localized structure.

A possible choice results from making

$$\varepsilon_t = 0 \tag{35}$$

leading to

$$\varepsilon = -ik'\ell\,, \tag{36}$$

giving

$$\begin{cases} \varepsilon_i = k\ell - ik'\ell \\ \varepsilon_r = k\ell + ik'\ell \end{cases} \tag{37}$$

For this case, the function (29) in the tunneling region assumes the form

$$e^{\frac{(ik'(x-\ell)-\omega t+\varepsilon_t)^2}{\alpha^2}} = e^{-\frac{(x-\ell)^2}{\alpha^2/k'^2}}\; e^{\frac{\omega^2 t^2}{\alpha^2}}\; e^{-2ik'(x-\ell)\omega t}\,. \tag{38}$$

In the case of the normal region, we have from (28) the incident amplitude

$$e^{\frac{(k(x-\ell)-\omega t+\varepsilon_i)^2}{\alpha^2}} = e^{\frac{(k(x-\ell)-\omega t+(k\ell-ik'\ell)_i)^2}{\alpha^2}} = e^{\frac{(k(x-\ell)-\omega t-ik'\ell)^2}{\alpha^2}}\,, \tag{39}$$

and for the reflected

$$e^{\frac{(k(x-\ell)+\omega t+\varepsilon_i)^2}{\alpha^2}} = e^{\frac{(k(x-\ell)+\omega t+(k\ell+ik'\ell)_i)^2}{\alpha^2}} = e^{\frac{(k(x-\ell)+\omega t+ik'\ell)^2}{\alpha^2}}\,. \tag{40}$$

Finally, looking at (19), (23) and (38) we are allowed to write for the wave in the tunneling region

$$\psi_2(x,z,t)=\frac{2k}{k-ik'}Ae^{-\frac{(x-\ell)^2}{\alpha^2/k'^2}-k'x}\,e^{\frac{\omega^2t^2}{\alpha^2}}\,e^{-2ik'(x-\ell)\omega t}\,e^{i(k_1\sin\theta_1 z-\omega t)}\,, \tag{41}$$

giving for the intensity

$$I_2=\frac{4k^2}{k^2+k'^2}\,|A|^2\,e^{-\frac{(x-\ell)^2}{\alpha^2/2k'^2}-2k'x}\,e^{\frac{\omega^2t^2}{\alpha^2/2}},\quad x\geq 0\,. \tag{42}$$

Rescaling A such that

$$A=A'e^{k'\ell}\quad\rightarrow\quad |A|^2=|A'|^2\,e^{2k'\ell} \tag{43}$$

formula (42) becomes

$$I_2=\frac{4k^2}{k^2+k'^2}\,|A'|^2\,e^{-\frac{(x-\ell)^2}{\alpha^2/2k'^2}-2k'(x-\ell)}\,e^{\frac{\omega^2t^2}{\alpha^2/2}},\quad x\geq 0\,, \tag{42'}$$

which maximum

$$I_{2M}=\frac{4k^2}{k^2+k'^2}\,|A'|^2\,e^{\frac{\omega^2t^2}{\alpha^2/2}} \tag{44}$$

along the *xx* axis, is centered at the point $x=\ell$.

These facts, clearly show that starting from an incident wave of the form, see (19) and (43),

$$\psi_{1I}(x,z,t)=A'e^{k'\ell}\,e^{\frac{(kx-\omega t-ik'\ell)^2}{\alpha^2}}\,e^{i(kx+k_1\sin\theta_1 z-\omega t)} \tag{45}$$

or,

$$\psi_{1I}(x,z,t)=A'e^{k'\ell-k'^2\ell^2/\alpha^2}\,e^{\frac{(kx-\omega t)^2}{\alpha^2}}\,e^{-i((2k'\ell(kx-\omega t)-k_1\sin\theta_1 z)} \tag{45'}$$

it is possible to obtain (42) a gaussian structure localized at a distance ℓ from the origin.

$$I_2=\frac{4k^2}{k^2+k'^2}\,|A'|^2\,e^{-\frac{(x-\ell)^2}{\alpha^2/2k'^2}-2k'(x-\ell)}\,e^{\frac{\omega^2t^2}{\alpha^2/2}},\quad x\geq 0$$

Since this parameter ℓ is as large as one wish, this means that it is actually possible to place a localized structure, in practically no time, at any point we like. Incidentally, because the position of the localized structure does not depend on time, the theoretical ideal transmission is, in this approach, instantaneous.

In order to make the plot of the intensity distribution in the two regions it is convenient to calculate the total intensity in the normal region. From the form of the incident and reflected waves, see (19) and (22) we can write

$$I_1 = |\psi_1|^2 = A\left[e^{(kx-\omega t - ik'\ell)^2/\alpha^2 + ikx} + \frac{k-ik'}{k+ik'} e^{(kx+\omega t+ik'\ell)^2/\alpha^2 - ikx} \right] e^{i(k_1 \sin\theta_1 z - \omega t)} \times$$
$$A^*\left[e^{(kx-\omega t+ik'\ell)^2/\alpha^2 - ikx} + \frac{k+ik'}{k-ik'} e^{(kx+\omega t-ik'\ell)^2/\alpha^2 + ikx} \right] e^{-i(k_1 \sin\theta_1 z - \omega t)} \tag{44}$$

which, after some calculations, give

$$I_1(x,t) = 2|A|^2 \, e^{(k^2x^2+\omega^2t^2-k'^2\ell^2)/\alpha^2/2} \times$$
$$\left[\mathrm{ch}\left(4\frac{k\omega}{\alpha^2} x t\right) + \frac{k^2-k'^2}{k^2+k'^2} \cos\left[2\left(\frac{2k'\ell}{\alpha^2}-1\right)k x\right] + \frac{2kk'}{k^2+k'^2} \sin\left[2\left(\frac{2k'\ell}{\alpha^2}-1\right)k x\right]\right], \quad x \le 0 \tag{45}$$

An approximate plot of the intensity along the *xx* axis, at a time *t=0*, is shown in Fig.5.

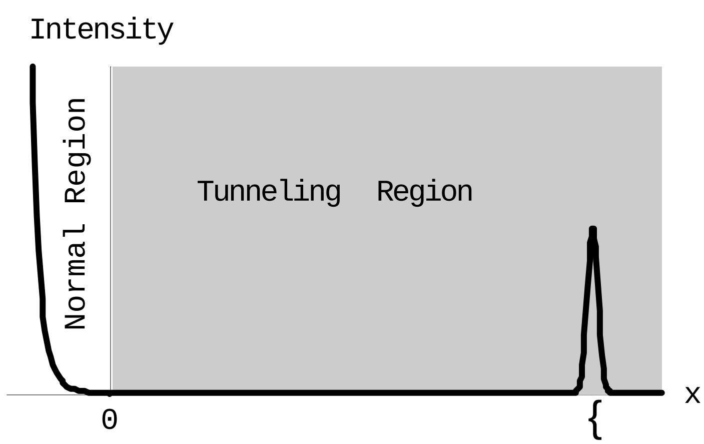

Fig.5 – Approximate representation of the intensity distribution in the two regions for the time *t=0*.

6 – Conclusion

These theoretical results seem to indicate that in tunneling conditions the usual relativistic velocity limit $\upsilon \le c$ breaks down allowing practically instantaneous motion. The problem now is to find out if these results are more than mere theoretical conclusions and have indeed any real physical correspondence.

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