**Free course:** ***Three Revolutions in Geometry***

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Presentation of sessions and literary references

<u>Session 1</u> *(30.04.2012): "A first «revolution» in mathematics: the transition from a single, plan space, to a plurality of curved, multidimensional spaces (Riemann, Klein, Poincaré)"*

In this first session, we will question the status of geometry as it evolved from the second half of the 19th century to the early 20th, that is, between the works by Riemann, Clifford, Beltrami, Helmholtz, Klein, Lie and Poincaré, and those by Hilbert, Cartan and Weyl. During this period, geometry knows the undoubtedly more fundamental transformation of its history, both in what concerns its methods and its concepts. The relationship of geometry with other branches of mathematics, especially algebra and analysis, will be deeply changed, as well as its relationship with physics and with other natural sciences. Therefore we no longer speak of geometry nor of space, but of *geometries* and of *spaces*. The recognition of a plurality of geometries in the mathematical plan constituted a capital historical fact. For the first time, with the discovery of non-Euclidean geometries, the conception of a single geometry and of an absolute space is completely questioned, in favor of another, radically different: as science of «pure» forms, geometry belongs to mathematics at the same level as arithmetic and algebra; while as science of real forms, it is intimately linked to physics. Riemann was undoubtedly the first to mathematically expose the double nature of space. In his notable 1954 dissertation, "*On the hypothesis which underlie geometry*" ("*Über die Hypothesen welche der Geometrie zu Grunde liegen*"), the author introduces completely new mathematical ideas, whose philosophical value and the meaning to physics appear revolutionary for the time. Following Gauss, but generalizing considerably his intuitions, Riemann shows that the Euclidean space, from a purely mathematical point of view, was no more than a particular case among other possible spaces, and that there was no reason to think that the physical space corresponded to the one described by the axioms of Euclidean geometry. Consequently, there could exist not only several geometries, but also several geometrical spaces (kinds of manifolds) and several different physical spaces. It was certainly a turning point which deeply changed the landscape of mathematics, but mathematics as well. In the presentation, we will analyze the steps that drove to this new conception and to the mathematical ideas which are its ground. Particularly, we will analyze the way in which the concept of *manifold* (*Mannigfaltigkeit*) in Riemann was formed and the way in which the modern differential geometry was constituted. The geometrical concept of manifold maintains an essential link with the functional concept of «Riemann surface», both of which can be explained thanks to a *qualitative* or *spatial* conception of mathematics. We will also provide an epistemological interpretation of the concept of manifold (*n*-dimentional curved space) and of its meaning. Finally, we will show that Riemann's manifold concept and Clifford's spatial theory of matter are at the basis of a fruitful movement of geometrization of physics, which culminated in Einstein's general theory of relativity.

**References**:


Boi, L., Flament, D., Salanskis, J.-M., *1830-1930: A Century of Geometry. Epistemology, History and Mathematics*, Springer-Verlag, "Lecture Notes in Physics", Heidelberg, 1992.

—, *Le problème mathématique de l'espace. Une quête de l'intelligible*, preface by R. Thom, Springer, Berlin, 1995.

—, "Nouvelles dimensions mathématiques et épistémologiques du concept d'espace en physique, de Riemann à Weyl et à Witten", in *L'espace physique entre mathématiques et philosophie*, M. Lachièze-Rey (éd.), EDP Sciences, Paris, 2006, pp. 101-133.

—, "La géométrie: clef du réel? Pensée de l'espace et philosophie des mathématiques", *Philosophiques*, 24 (2), 1997, pp. 389-430.

—, "Mannigfaltigkeit und Gruppenbegriff. Zur den Veränderungen der Geometrie im 19. Jahrhundert", *Mathematische Semesterberichte*, 41 (1), 1994, pp. 1-16.

—, "From Riemannian Geometry to Einstein's General Relativity Theory and Beyond: Spacetime Structures, Geometrization and Unification", in *Albert Einstein Century International Conference Proceedings*, Conference Proceedings Series 861, J.M. Alimi and A. Füzfa (eds.), American Institute of Physics Publishers, Melville, pp. 1066-1075.

*<u>Session 2</u> (02.05.2012): "A second «revolution» in mathematics: the new interaction between geometry and physics, from a pre-determined space to a dynamic space-time (Clifford, Minkowski, Einstein, Weyl)"*

The existence of several geometries which are also carried out from the point of view of physics (other than the mathematical), was shown in a decisive way thanks to Einstein's general theory of relativity, even though especially Riemann and Clifford had already admitted that a geometry other than the Euclidean could be applied to our physical space. But to come to such admission, it was first necessary to deeply criticize our conception of space, which could no longer be thought nor as the place where figures can be constructed, nor as the one where bodies move. In his fundamental work about the hypothesis of geometry, Riemann showed that the property of continuity is linked to the metrical structure of space, which means that each point, as well as its infinitesimal variations, is representable by a continuous function of its differentials. Moreover, he demands such functions to be continually differentiable, which defined the *differentiable* level of the continuum, after he had recognized the existence of a first topological level of continuity, which could be designated by that of *dimensionality* – which can equally be expressed by saying that the world we inhabit is a spatial continuum of three dimensions (or a tridimensional manifold). But Riemann sets forth the possibility that there is a third level of the continuum, whose nature isn't at all assimilable to the others we have just mentioned, and whose constitutive principles do not take part in the way we abstractly represent them, as is the case to the discrete (the discrete manifolds, like the arithmetic or algebraic manifolds, are composed by numerable elements, while the continuous manifolds composed by points are measurable through functions of distance). According to Riemann, continuous manifolds (like the differentiable manifolds) could have an origin of dynamic nature, that is, the property of continuity would be linked to the physical content of space. In other words, the physical phenomena and the kind of space in which they take place are indissociable: the space imagined by Riemann is non-empty (differently from the one thought by Newton) and endowed with physical effects which would propagate locally. Following Riemann, Clifford explicitly theorized a coherent program for a geometric interpretation of the physical phenomena. Clifford takes over Riemann's intuition and states the hypothesis that physical space (both at macroscopic as microscopic scale) isn't neither homogeneous nor isotrope, that is curved and not flat (as in Euclidean geometry) and susceptible of variation under the presence of certain physical effects, and that, furthermore, the behavior of matter depends on how the curvature of space varies. We will show that the spatial theory of curvature and of matter developed by Clifford will play an important role in the development of general relativity until a recent period, particularly in the elaboration of J. A. Wheeler's geometrodynamic theory. Riemann's influence (and Clifford's indirectly) upon the new physics, and particularly on Einstein's, has been tremendous. Actually, the latter's general relativity is grounded on the concept of Riemann's differentiable manifold, which was endowed with a non-Euclidean metric (hyperbolic, elliptic or other), and with complicated geometrical objects which we call curvature tensors. These are geometrical objects which also have a physical meaning, provided that they correspond to the gravitation potential of general relativity. That means, in other terms, that the properties of the phenomena which occur at the scale of our Universe lie in the geometrical and topological structure of a pseudo-riemannian manifold, Einstein's space-time constituting its physical model par excellence.

**References**:

Boi, L., "Theories of space-time in modern physics", *Synthese*, 139 (2004), pp. 429-489.

—, "Géométrie de l'espace-temps et nature de la physique: quelques réflexions historiques et épistémologiques", *Manuscrito,* 23 (1), 2000, pp. 31-98.

—, "Geometrical and topological foundations of Theoretical physics: from gauge théories to string program", *International Journal of Mathematics and Mathematical Sciences*, 34 (2004), 1777-1836.

—, "Ideas of Geometrization, Geometric Invariants of Low-Dimensional Manifolds and Topological Quantum Field Theories", *International Journal of Geometric Methods in Modern Physics*, 6 (4), 2009, 1-54.

—, "Geometria e dinamica dello spazio-tempo nelle teorie fisiche recenti. Su alcuni problemi concettuali della fisica contemporanea", *Giornale di Fisica*, Società Italiana di Fisica, 50 (1), 2009, 1-10.

—, "Clifford Geometric Algebras, Spin Manifolds, and Group Action in Mathematics and Physics", *Advances in Applied Clifford Algebras*, 19 (3), 2009, 1-42.

<u>Session 3</u> *(04.05.2012): "A third «revolution» in mathematics: geometry and the generation of natural and perceptive forms (D'Arcy Thompson, Thom, neo-gestalt)*"

Catastrophe theory elaborated by René Thom in the 1960's is a domain of differential topology and it is part of the mathematical theory of singularities of differentiable applications, founded by the mathematician H. Whitney and developed after by John Mather. The theory of singularities is a generalization of the functions' minima and maxima. Whitney replaces the functions by mappings, that is, collections of multiple functions with several variables. The bifurcations of the dynamic systems of A. Andronov, already introduced by Henri Poincaré at the beginning of the last century in the framework of his celestial mechanics and of systems of chaotic type, constitute one of the essential mathematical ingredients of catastrophe theory. Bifurcations are geometrical objects characterized by an unstable behavior; places where a function ceases to be linear and acquires further determinations. In a broader sense, the word bifurcation designates every kind of qualitative reorganization or metamorphosis of different entities resulting from a change of parameters of which they depend. Catastrophe theory aims to describe the discontinuous phenomena with the support of continuous mathematical models. In other words, the theory has as its goal to build continuous dynamic models as simple as possible which can generate morphologies, empirically given, or sets of discontinuous phenomena. We will show that the notion of *singularity* is one of the most fundamental in mathematics; it is at the heart of the theory of functions with several complex variables, of Riemann's surface theory, of algebraic geometry, of differential topology, and certainly, of the qualitative theory of dynamic systems. The singularities of the function are in a sense traces of the topology that we «killed»: we kill the manifold's topology by applying it the real axis, but the topology resists, it «screams», and its cries are manifested by the existence of critical points. Hence the notion of singular point, which plays a most fundamental role in catastrophe theory. Generalized catastrophe theory's basic postulate is that the form under which every object appears to the observer is nothing more than the set of catastrophes associated to a certain dynamic. Thus the boundary which separates it from the external medium, in regions where it doesn't display accidents, will be frequently associated to a *fold*-type catastrophe. But it can dig a trench, exfoliate in a bubble, issue a lash, in which cases one as to appeal to the *cusp*, to the *swallowtail*, to the *elliptic umbilic*. Several complex situations might be found, only describable by the generalized catastrophe theory: bubbles, lumps, laminars, filaments. They are part of our everyday landscape, to the point that we no longer pay them attention: it's the foam of a glass of beer, the condensation of a cloud in rain, the cracks in an old wall, the drawings left by the wave in the sand. There is a sort of ideal geometric reality in the dynamic systems studied by Thom, and that's probably the reason (or one of the reasons) why the form of a wave crashing in the seashore evokes, irresistibly, the hyperbolic umbilic. But these dynamic systems (natural phenomena, living processes) also have, simultaneously, a content of sense, meaning which, through the mediation of certain physical qualitative resonances (*prégnances*) (light, sound, heat, etc.) that invest the observer's field by reshaping his centers of awareness and by causing variations in his points of attention regarding objects and events. We will show that catastrophe theory is, first of all, a theory of action, a dynamic theory of possible developments of forms. In other words, catastrophe theory is interested in the formulation of a dynamic theory of generation, of becoming, and of stabilization of natural and living forms.

**References:**

Boi, L., *Morphologie de l'invisible. Transformations d'objets, formes de l'espace, singularités phénoménales et pensée diagrammatique,* Presses Universitaires de Limoges, PULIM, 2011.

—, "Geometry of dynamical systems and topological stability : from bifurcations, chaos and fractals to dynamics in the natural and life sciences", *International Journal of Bifurcation and Chaos*, 21 (3), 2011, pp. 815-867.

—, "Images et digrammes des objets et leurs transformations dans l'espace", *Visible*, 5 (2009), pp. 77-109.

—, "Topological ideas and structures in fluid dynamics", *JP Journal of Geometry and Topology*, 8 (2), 2008, pp. 151-184.

—, "Topological Knot Theory and Macroscopic Physics", in *Encyclopedia of Mathematical Physics*, J.-P. Françoise, G. Naber, T.S Sun (eds.), Elsevier, Oxford, 2006, pp. 319-327.

—, *Symétries, Brisures de Symétries et Complexité, en mathématiques, physique et biologie*, Peter Lang, Bern, 2006.