The Twin Paradox, Once More

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Abstract

The so-called twin paradox in special relativity, first introduced by Paul Langevin, aims to present, in a specially cogent manner, the difference between the aging rates of two twin brothers as one stands still in some inertial reference frame - the Earth frame will do, for this purpose - while the other executes a fro and back trip in two legs, at a constant speed in each one of them, with respect to the brother at rest on Earth. As is well known, much confusion came out of this situation, but a especially clear, recent discussion, by the philosopher Tim Maudlin, stimulated the following note that aims to put forth clearly the mistake contained in one very common explanation.

1 The case of two twins

Following the discussion in Maudlin [1], we will use only the invariance of the relativistic interval between any two events in our discussion, without recourse to the Lorentz transformations. In this manner, the geometrical characteristics of the Minkowski metric of spacetime are all that is needed to put forward the argument.

In Minkowskian special relativity, one considers occurrences or events as the building elements of all physical phenomena. Examples would be collisions between bodies, reflections of light rays on surfaces, and so on. These events take place in space and time and, once one adopts some appropriate inertial reference frame R, one can attach definite coordinates to each one of them. For the present purposes, this common approach should be enough, but a deeper discussion can be followed in Maudlin's book. Then, following Minkowski, one defines the relativistic interval Δs between any two events E_1 and E_2, with coordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2), by the positive solution of the equality

$$\Delta s = \sqrt{c^2(t_2 - t_1)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2}, \qquad (1)$$

a result expressed in meters, naturally ($c = 3 \times 10^8$ m/s being the speed of light in vacuum). The geometry of Minkowski spacetime is such that the number obtained for the interval between any two events is independent of the reference frame in which their respective coordinates are being measured. That is to say, spacetime intervals are relativistic invariants, if $\Delta s'$ is associated with the prime coordinates from another frame R', then $\Delta s = \Delta s'$. In particular, one can choose the reference frame attached to some physical object, its proper frame (such as one of the twin brothers), and then, not only is the difference in spatial coordinates between any two events that happen to that body zero (we are considering twins reduced to single mathematical points !), but also the time lapse between those same events has now a special meaning, being called the "proper time" between those events, traditionally symbolized by the greek letter τ. Consequently, one measures the interval $c\Delta\tau$ in the proper frame (with $\Delta x = \Delta y = \Delta z = 0$), and the same value in any other inertial frame even though the spatial distances are no longer zero, that is to say, $c\Delta\tau = \Delta s$.

Let us now consider the Minkowskian spacetime diagram in the figure below. For simplicity, only the x space coordinate is depicted, along the horizontal axis, and the time, also measured in meters through the product ct, runs along the vertical axis. This is the frame attached to twin A, its proper frame. As time goes by, twin A occupies successive positions along the positive vertical axis, always at $x = 0$ (the green line). Twin B, on his part, starts moving along the positive x-axis of twin A's frame, and his successive positions are depicted on the Minkowski diagram as occupying the red line from the origin $(0, 0)$ to spacetime point $(5, 4)$, i.e., reaching the $x = 4$ m coordinate of this frame at the instant $ct = 5$ m as measured in twin A's frame, and then from this point in spacetime to point $(ct = 10, x = 0)$, that is to say, back to the origin, meeting twin A again (here we are using exactly the same numerical example as in Maudlin's).

Now, given the metric properties of Minkowski spacetime, the interval between the two events, of twin A seeing his twin B leave and having him back again will be

$$\Delta s \mid^{A}_{(0,0)\to(10,0)} = \sqrt{(10-0)^2-(0-0)^2} = 10 = c\Delta\tau \tag{2}$$

as measured by twin A. So, A aged $\Delta\tau = 10/c$ seconds between these two events.

Between these same two events, labeled as $(0,0)$ and $(10,0)$ in this frame, the total interval measured with respect to twin B, which occupies coordinate $x = 4$ at the intermediate time $ct = 5$, is given by the sum of the intervals obtained for each leg:

$$\Delta s \mid^{B}_{total} = \Delta s \mid^{B}_{(0,0)\to(5,4)} + \Delta s \mid^{B}_{(5,4)\to(10,0)} . \tag{3}$$

Substituting for the concrete values in the example, we get

$$\Delta s \mid^{B}_{total} = \sqrt{(5-0)^2-(4-0)^2} + \sqrt{(10-5)^2-(0-4)^2} = 6, \tag{4}$$

there is to say, the time lapse between the two events, as measured by the B twin in his own frame (not used above) but using the data above from A's frame, will be $\Delta s \mid^{B}_{total} = 6/c = \Delta\tau$ seconds, less the one corresponding to twin A, $10/c$.

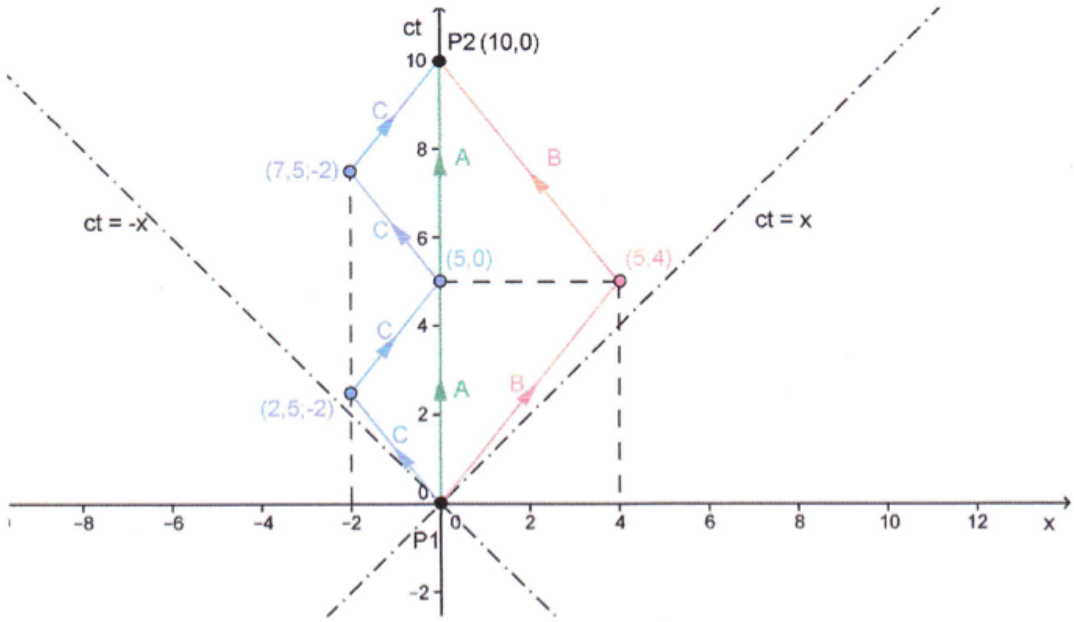
2 Mistaken explanations

Maudlin [1] considers three kinds of confusion surrounding the explanation of the twin paradox. The first one claims that the effect can not occur because it would represent a breach of the supposed total equivalence between reference frames ("all motion is relative"), which is in fact false. The second confusion, perhaps the most common, argues that what distinguishes the two twins A and B is the acceleration felt unequivocally by twin B; this different acceleration would by itself imply the different agings (he cites both Rindler [2] and Feynman [3], in this regard, but, in fact, the examples are endless). As a matter of fact, what matters, as Maudlin points out, is the length of the trajectories in the spacetime diagrams, and not their particular shape (in this regard, see also reference [4], page 125 : "proper time is not integrable"). It is this second confusion that matters to us, presently. Finally, there is the confusion about the supposed effects of "speed" at which some clock moves on the time it gives, whereas, in fact, speed is irrelevant.

As just mentioned, what matters here is the common argument according to which it is the fact that twin B suffered an acceleration at the intermediate position, here ($ct = 5, x = 4$), that justifies him aging less, establishing a causal connection of the kind *different accelerations* $\Rightarrow$ *different agings*.

3 Unequal accelerations, equal agings

We consider, then, the common explanation that invokes the fact that the two twins suffer different accelerations and that it is this circumstance that ultimately justifies their different agings. Now, if it were true that different accelerations imply a brake up of symmetry and thus different agings, then, by *modus tollens*, equal agings should necessarily imply having suffered equal accelerations. That this is not so will be clear from an analysis of the diagram. We take again the two twins, A, B and, as before, A stays put at $(ct, x = 0)$ starting at $(ct = 0, x = 0)$, whereas B moves from the origin $(ct = 0, x = 0)$ to spacetime point $(ct = 5, x = 4)$ on the first leg, and from this to spacetime point $(ct = 10, x = 0)$ on the second leg, i.e., back to the origin.



But now we consider a third twin brother, C, who performs a more crooked trajectory in spacetime, leaving the vicinity of twin A at the same time and place as twin B (i.e., at the spacetime origin $(0, 0)$), but moving in the negative direction along the x axis at a constant speed until he reaches the spacetime point $(ct = 2.5, x = -2)$ where he inverts his unidimensional trajectory and returns to the space origin $x = 0$, meeting twin A at spacetime point $(ct = 5, x = 0)$. Here he rebounds once more, moves along the negative x axis to point $x = -2$ again, reaching it at time $ct = 7.5$, and then back to the position of twin A, at $x = 0$, which he reaches at time $ct = 10$ (the blue line in the diagram).

Let us now compute the total relativistic interval for twin C. We have

$$\Delta s \mid^{C}_{total} = \Delta s \mid^{C}_{(0,0)\to(2.5,-2)} + \Delta s \mid^{C}_{(2.5,-2)\to(5,0)} + \Delta s \mid^{C}_{(5,0)\to(7.5,-2)} + \Delta s \mid^{C}_{(7.5,-2)\to(10,0)} \tag{5}$$

$$= \sqrt{(2.5-0)^2 - (-2-0)^2} + \sqrt{(5-2.5)^2 - (0+2)^2} + \sqrt{(7.5-5)^2 - (-2-0)^2} + \sqrt{(10-7.5)^2 - (0+2)^2} = 6$$

in meters. One can see that this is exactly the same value obtained for the relativistic interval in the case of twin B.

4 Conclusion

The obvious conclusion form the above example is that, in spite of having performed different motions with different accelerations, twins B and C suffered exactly the same aging, and both aged less than twin A. Consequently, different accelerations can not justify differences in aging.

References

[1] Tim Maudlin, *Philosophy of Physics - Space and Time*, Princeton University Press (2012).

[2] Wolfgang Rindler, *Essential Relativity*, Springer (1977).

[3] R. Feynman, R. Leighton and M. Sands, *The Feynman Lectures on Physics, vol. I, p. 16-3*, Addison-Wesley (1963).

[4] R. Adler, M. Bazin, M. Schiffer, *Introduction to General Relativity*, McGraw-Hill (1975).