Computational Models of Measurement and Hempel's Axiomatization

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Abstract. We have developed a mathematical theory about using physical experiments as oracles to Turing machines. We suppose that an experiment makes measurements according to a physical theory and that the queries to the oracle allow the Turing machine to read the value being measured bit by bit. Using this theory of physical oracles, an experimenter performing an experiment can be modelled as a Turing machine governing an oracle that is the experiment. We consider this computational model of physical measurement in terms of the theory of measurement of Hempel and Carnap (see [16, 13]). We note that once a physical quantity is given a real value, Hempel's axioms of measurement involve undecidabilities. To solve this problem, we introduce time into Hempel's axiomatization. Focussing on a dynamical experiment for measuring mass, as in [1, 3, 5, 4, 6], we show that the computational model of measurement satisfies our generalization of Hempel's axioms. Our analysis also explains undecidability in measurement and that quantities are not always measurable.

1 Introduction

We are developing a methodology and mathematical theory to examine how data is represented and computations are performed by physical systems. The research programme is shaped by questions about what can be computed by (i) physical systems in isolation and (ii) physical systems combined with algorithms. The methodology is formulated using

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five principles that focus on the role of a physical theory in formalising experiments. Our theory for isolated physical systems begins in [8–11] and that for physical systems and algorithms begins in [1–4]. A central technical idea is to use a physical experiment as an oracle to a Turing machine. This changes the nature of oracle queries and introduces new and subtle *protocols* to manage the time taken by queries and tolerances in data exchanges. Typically, we use an experiment $E(x)$ designed to measure a physical quantity represented by a real number x. The oracle is expected to extend the computing power of the Turing machines. For specific experiments, we have characterised the class of sets decidable by these machines using non-uniform complexity classes and we have shown that the oracles extend the power of Turing computability substantially.

However, recently in [4, 6], we have added a new, sixth principle which changes the perspective of the mathematical theory of Turing machines with physical oracles. Instead of viewing the experiment as an oracle boosting the power of Turing machines, we view the Turing machine as controlling and, indeed, performing the experiment. Specifically, Principle 6 leads us to suppose that:

The Turing machine models a human experimenter conducting the experiment.

The relationship between experimenter and experiment is modelled by the protocols that apply to the oracle queries. In [4] we study in some detail a Newtonian experiment to measure mass, which reveals concepts and properties of wide applicability.

Thus, with Principle 6 of [4, 6], we find we are in possession of a fledgling computational model of the process of doing physical experiments and making measurements. The model accommodates

(i) logical properties of the process of following an experimental procedure, made up of instructions specified by a physical theory; and

(ii) quantitative constraints of precision and error margins and of the cost in time and other resources needed to perform experiments.

We have looked at several experiments and the questions arise:

To what extent is our computational model of experimentation general? What is measurement?

In this paper we begin to explore these questions with the help of the philosophy of physics. We relate our computational model to the desiderata of Geroche and Hartle [15] for an investigation into computable aspects to measurement. We consider the axiomatic theory of measurement established by Carl G. Hempel in [16], and elaborated by Rudolf Carnap in [13], and apply it to our computational models of measurement. *Do our* *models satisfy Hempel's axioms?* Yes. *Do they reveal new general properties of measurement?* Yes. Indeed, we show that the models uncover some shortcomings in Hempel's characterisation, which we repair with new axioms.

Hempel's theory is based on two predicates intended to make *comparisons between some physical attribute*: think of an equivalence and ordering applied to some attribute of a set of objects. On measuring the attribute using real numbers, the comparison predicates are mirrored by the standard predicates $=$ and $<$, which are undecidable on computable real numbers. This is more than an inconvenience for an axiomatic theory of measurement, where tolerances and accuracy are central concepts. This undecidability can be ameliorated in different ways. We introduce the operational concept of computational resources, specifically *time*, into Hempel's axioms; the resulting axiomatisation we believe to be new. The idea of considering time as a cost in deciding the equality of measurements is suggested by our previous technical work on the model (e.g., see [1, 3]).

Let us consider the impact of adding time to Hempel's view of measurement. Hempel uses the experience of measuring mass with a balance scale to introduce his axioms. The notions of two objects weighing the same, or one weighing less than the other, are quite intuitive. However, as the masses of the two objects approach one another, the measurement becomes more and more troublesome, due to friction and nature of the balance: *two objects in the pans may be in equilibrium one day but are found no longer to be in equilibrium the next.* Hempel in [16], end of Chapter 10 and middle of Chapter 11, develops the following argument:

Hempel 1 The most important — and perhaps the only — type of fundamental measurement used in the physical sciences is illustrated by the fundamental measurement of mass, length, temporal duration, and a number of other quantities. It consists of two steps: first, the specification of a comparative concept, which determines a nonmetrical order; and, second, the metrization of that order by the introduction of numerical values [...] Now we return to our illustration [of measuring mass]. In formulating specific criteria for this case, we will use abbreviatory phrases: of any two objects, x and y [...] we will say that x *outweights* y if, when the objects are placed into opposite pans of a balance in a vacuum, x sinks and y rises; and we will say that x *balances* y if under the conditions described the balance remains in equilibrium.

Hempel is aware of the need of improving accuracy to define metrical properties for the mass concept (hence the vacuum[5]). However, there is

[5] Why should the balance be in a vacuum? It is not because of friction. It is because there are substances in the atmosphere that have "negative weight" such as hidrogenium and helium.

no awareness, either in Hempel's or in Carnap's theories, that the *time to run an experiment* is actually a fundamental concept when allocating numerical values to attributes in a consistent way. Hempel is conscious of this limitation of his axiomatization of measurement of quantities that take real values, or even rational values. In a footnote, he declares the following:

Hempel 2 This account of the fundamental measurement of mass is necessarily schematized with a view to exhibiting the basic logical structure of the process. We have to disregard such considerations as that the equilibrium of a balance carrying a load in each pan may not be disturbed by placing into one of the pans an additional object which is relatively light but whose mass is ascertainable by fundamental measurement. This means that fundamental measurement does not assign exactly one number to every object [...]

Measurement is a mapping from objects to numbers. By introducing time in Hempel's axiomatization, we establish a more accurate semantical basis for the these maps.

The structure of the paper is this. In Section 2, we review the Hempel-Carnap theory of measurement. In Section 3, we recall the computational model of an experiment to measure mass from [4]. Such computational models are *gedankenexperimente*. We review the ideas of Geroch and Hartle [15] in Section 4. In Section 5 we look at mass in Newtonian dynamics. In Section 6, we present a new axiomatization of measurement by generalising Hempel's axioms in order to introduce the *time taken by a measurement process*. This is, indeed, a generalisation, from which we can recover the old axiomatization. Finally, in Section 7, we show how the computational perspective implies that not all quantities are measurable.

2 Theory of measurement

2.1 The three concepts of measurement

According to Hempel in [16], and Carnap in [13], the construction of a quantitative concept, based on measurement, involves three phases. For illustration, we use the quantitative concept of mass as measured by the balance.

The classificatory phase. Classification is based upon some primitive method of sorting concepts into groups according to similarities. What aspect is chosen is termed an *attribute*. Classification is essentially subjective. To make finer classifications, attention must be paid to details of the objects being classified, which demands more time of the taxonomist.

The comparative phase. The attributes that define the classification need to be compared. A comparative concept is something observable of attributes and what is observed is termed an *event*. It constitutes the basis for a quantitative concept; although the comparative concept seems to be unique, the quantitative one can be understood and axiomatized in different ways.

For the concept of weight, we introduce the comparative concepts of *lighter*, *heavier*, and *equal* in weight. These concepts have an empirical procedure by which we can take any pair of objects and observe:

If the two objects balance, they are of equal weight. If the objects do not balance, the object on the pan that rises is lighter than the object on the pan that sinks.

Let these observable events define the relations of "equality" $\mathcal{E}$ and "less than" $\mathcal{L}$, respectively.

The quantitative phase. The attributes we wish to compare are assigned numerical values by a map M from objects to numbers. In [13], Carnap says:

Carnap 1 The qualitative language is restricted to predicates (for example, "grass is green"), while the quantitative language introduces what are called functor symbols, that is, symbols for functions that have numerical values. This is important, because the view is widespread, especially among philosophers, that there are two kinds of features in nature, the qualitative and the quantitative. Some philosophers maintain that modern science, because it restricts its attention more and more to quantitative features, neglects the qualitative aspects of nature and so gives an entirely distorted picture of the world. This view is entirely wrong, and we can see that it is wrong if we introduce the distinction at the proper place. When we look at nature, we cannot ask: "Are these phenomena that I see here qualitative phenomena or quantitative?" That is not the right question. If someone describes these phenomena in certain terms, defining those terms and giving us rules for their use, then we can ask: "Are these the terms of a quantitative language, or are they the terms of a prequantitative, qualitative language?".

The measurements must preserve the comparisons. For mass, we need to define the relations between the events associated with the balance scale and the map M: for any objects a and b, (i) if $a\mathcal{E}b$ then $M(a) = M(b)$ and (ii) if $a\mathcal{L}b$ then $M(a) < M(b)$.

2.2 The axiomatization of measurement

In Hempel's book [16], Part III, Chapters 9 to 13, we find an axiomatization of measurement in Physics and other empirical sciences; a discussion of Hempel's axiomatization is Carnap [13].

Consider a class $\mathcal{O}$ of physical objects endowed with some attribute (such as *mass*, *electric charge*, or *temperature*, etc.). A measurement of an attribute in the sense of Hempel is a map $M : \mathcal{O} \to N$, where N is a number system such as the *integers* $\mathbb{Z}$, *rationals* $\mathbb{Q}$, or *reals* $\mathbb{R}$. For definiteness, we will choose $M : \mathcal{O} \to \mathbb{R}$.

Hempel's axiomatization of measurement establishes an ordering of the objects of $\mathcal{O}$. To have a measurement, we need an *instrument* or *experimental apparatus*, and observations defining events that implement physically the two special comparative predicates $\mathcal{E}$ and $\mathcal{L}$ over the set $\mathcal{O}$:

1. If objects a and b are identical in the observed attribute, then $a\mathcal{E}b$ is the case.
2. If object a is less than object b in the observed attribute, then $a\mathcal{L}b$ is the case.

The experimental apparatus works with the objects from $\mathcal{O}$, allowing the experimenter to establish a comparison of values of a given attribute.

Definition 1. *Given two binary relations $\mathcal{E}$ and $\mathcal{L}$, $\mathcal{L}$ is $\mathcal{E}$-*irreflexive *if, for all objects a and b in $\mathcal{O}$, if $a\mathcal{E}b$ is the case, then $a\mathcal{L}b$ does not hold.*

Definition 2. *Given two binary relations $\mathcal{E}$ and $\mathcal{L}$, $\mathcal{L}$ is $\mathcal{E}$-*connected *if, for all objects a and b in $\mathcal{O}$, if $a\mathcal{E}b$ does not hold, then $a\mathcal{L}b$ or $b\mathcal{L}a$ is the case.*

Definition 3. *Two binary relations $\mathcal{E}$ and $\mathcal{L}$ determine a* comparative concept*, or a* quasi-series*, for the elements of $\mathcal{O}$, if $\mathcal{E}$ is an equivalence relation and $\mathcal{L}$ is transitive, $\mathcal{E}$-irreflexive, and $\mathcal{E}$-connected.*

Let $\mathbb{E}$ be the set of observable events. Let $\mathcal{I} : \mathcal{O} \times \mathcal{O} \to \mathbb{E}$ be an *abstract implementation map.* In Hempel's examples in [16], the set $\mathbb{E}$ of events can be reduced to the bipolar set $\{-1, 0, +1\}$: the outcome of each experiment with objects a and b will tell us that either $a\mathcal{L}b$ (the event denoted by -1), or $a\mathcal{E}b$ (the event denoted by 0), or $b\mathcal{L}a$ (the event denoted by $+1$). The experimenter has to identify which physical events are to be denoted by $-1, 0, +1$.

In the example of the balance, if we put objects a and b in the left pan and the right pan, respectively. Event -1: the left pan rises and the right pan sinks — $a\mathcal{L}b$) is the case. Event $+1$: the left pan sinks and the right pan rises — $b\mathcal{L}a$ is the case. Event 0 (or the non-event): the balance remains in equilibrium — $a\mathcal{E}b$ is the case.

A careful reading of Chapter 12 of [16], on the notion of *fundamental measurement*, introduced by Campbell in [12], we find that a detailed substructure of $\mathcal{O}$ can be identified, consisting of a standard object, called the *unit mass*, together with its multiples and submultiples: this substructure we call the *toolbox of standards*[6]. By reducing the number of axioms in Hempel's theory (namely, removing the axioms of extensivity, developed by Suppes in [17]), we can provide a first workable definition of *measurement map* for a set of objects:

Definition 4. *Let $\mathcal{E}$ and $\mathcal{L}$ be comparative relations on the set $\mathcal{O}$ of objects (Definition 3). Suppose there exists an* experimental apparatus *to witness these relations and let $\mathbb{E}$ be a set of elements denoting* physical events.

Suppose $\{-1, 0, +1\} \subseteq \mathbb{E}$ and whenever the experiment is done with arbitrary objects $a, b \in \mathcal{O}$, if the outcome is event -1, then $a\mathcal{L}b$ is the case, if the outcome event is $+1$, then $b\mathcal{L}a$ is the case, and if the outcome is 0, then $a\mathcal{E}b$ is the case.

Then the map $M : \mathcal{O} \to \mathbb{R}$ is a measurement map *if*

Axiom 1 *If $a\mathcal{E}b$, then $M(a) = M(b)$.*
Axiom 2 *If $a\mathcal{L}b$, then $M(a) < M(b)$.*

We think this is a good definition capturing Hempel's construction of a quantitative concept from a comparative concept, as [16] suggests:

Hempel 3 Any function M which assigns to every element x of $\mathcal{O}$ exactly one real-number value, $M(x)$, will be said to constitute a *quantitative* or *metrical concept*, or briefly a *quantity* (with the domain of application $\mathcal{O}$); and if M meets the conditions just specified, we will say that it *accords with* the given quasi-series.

The axiomatization allows to prove simple results such as:

Proposition 1. *For all a, b in $\mathcal{O}$, one, and only one, of the following statements holds: (a) $a\mathcal{E}b$, (b) $a\mathcal{L}b$, or (c) $b\mathcal{L}a$.*

[6] This is done by considering a semigroup of objects $\mathcal{O} = \langle \mathcal{O}, \circ; 1 \rangle$, with the distinguished element 1 called the unit, and some internal structure to generate fractions and multiples of the unit.

Proof: First, we show that at least one of the three conditions hold. Suppose $a\mathcal{E}b$. Then we are done. Suppose that $a\mathcal{E}b$ is not the case. Since $\mathcal{L}$ is $\mathcal{E}$-connected, either $a\mathcal{L}b$ or $b\mathcal{L}a$. Thus, one of the three relations holds. We show that only one can hold.

(a) Suppose that $a\mathcal{E}b$. Since $\mathcal{L}$ is $\mathcal{E}$-irreflexive, $a\mathcal{L}b$ is not the case. Since $\mathcal{E}$ is an equivalence, $b\mathcal{E}a$ is also the case. Again, since $\mathcal{L}$ is $\mathcal{E}$-irreflexive, $b\mathcal{L}a$ is not the case.

(b) Suppose that $a\mathcal{L}b$. Since $a\mathcal{E}a$, we can not have $b\mathcal{L}a$, because by transitivity we would get $a\mathcal{L}a$ and $\mathcal{L}$ is $\mathcal{E}$-irreflexive. We can not also have $a\mathcal{E}b$, since $\mathcal{E}$-irreflexivity implies that $a\mathcal{L}b$, a contradiction.

(c) The argument is the same as (b). □

The converse of the axioms in Definition 4 hold.

Proposition 2.

$$\textit{If } M(a) = M(b), \textit{ then } a\mathcal{E}b. \tag{1}$$

$$\textit{If } M(a) < M(b), \textit{ then } a\mathcal{L}b. \tag{2}$$

Proof: We argue by contraposition. (1) Suppose that $a\mathcal{E}b$ is not the case. Then we have either $a\mathcal{L}b$ or $b\mathcal{L}a$, that is either $M(a) < M(b)$ or $M(b) < M(a)$, by definition. It follows that $M(a) \neq M(b)$. (2) Suppose now that $a\mathcal{L}b$ is not the case. Then either $a\mathcal{E}b$ or $b\mathcal{L}a$, that is either $M(a) = M(b)$ or $M(b) < M(a)$. □

Proposition 3.

$$\forall x \ \forall y \ (x\mathcal{E}y \ \Leftrightarrow \ \forall u \ ((x\mathcal{L}u \Leftrightarrow y\mathcal{L}u) \wedge (u\mathcal{L}x \Leftrightarrow u\mathcal{L}y))). \tag{3}$$

$$\forall x \ \forall y \ \forall z \ ((x\mathcal{E}y \wedge y\mathcal{L}z) \Rightarrow x\mathcal{L}z). \tag{4}$$

Axioms 4 and 4 in Definition 4, are not far from Hempel's own theory as stated in [16]:

Hempel 4 Let $\mathcal{E}$ and $\mathcal{L}$ be two relations which determine a quasi-serial order for a class $\mathcal{O}$. We will say that this order has been metricized if criteria have been specified which assign to each element x of $\mathcal{O}$ exactly one real number, $M(x)$, in such a manner that the following conditions are satisfied for all elements x, y of $\mathcal{O}$: [follows Axioms 4 and 4].

This (first) axiomatization of measurement[7] is troubled by the undecidability of $=$ for quantities ranging over the real numbers. In Section 6,

[7] There can be further structure for the map M, e.g., depending on the fact that the attribute considered is either *extensive* (e.g., mass) or *intensive* (e.g., temperature).

we will show how to generalize Hempel's axioms in order to have decidable comparison relations, by the introduction of time complexity to an experiment.

3 The collider experiment

In this section we describe an example of an experiment about elastic collision for the purpose of measuring the unknown (inertial) mass of a particle. The experiment is conducted exactly as described in [4]. This type of experiment to measure mass was and still is at the heart of mechanics. A generalization of the collision experiment can be used to measure the mass of a star or of a planet, measures that cannot be done with the balance scale.

3.1 Theory

As a *gedankenexperiment*, we consider a very simple situation at the limit of physical reality: a one dimensional elastic collision of two particles. The elastic collision between two particles on a line is dictated by two basic laws of Physics: the conservation of *linear momentum* and the conservation of *kinetic energy*, both of which can be derived from Newtonian laws of dynamics. See Section 5.

3.2 Experiment

In the one dimensional collision the center of mass of the two particles are in the same line of motion. Let m and μ be the masses of the two particles. We will assume that the particle of "unknown" mass μ is always at rest before the collision, and that the "proof" particle of mass m is projected along the line towards the particle of unknown mass μ with speed $u = 1.0\ (\pm\ \varepsilon)\ ms^{-1}$, e.g. with $0 \leq \varepsilon \leq 0.1$[8]. After the collision the particle of mass m acquires the speed v_m and the particle of mass μ is projected forward with speed v_μ.

By the conservation of momentum and kinetic energy, the collision is described by the equations:

$$mu = mv_m + \mu v_\mu, \tag{5}$$

[8] This error margin in the initial speed of the proof particle of mass m means that precision in speed does not matter for this experiment.

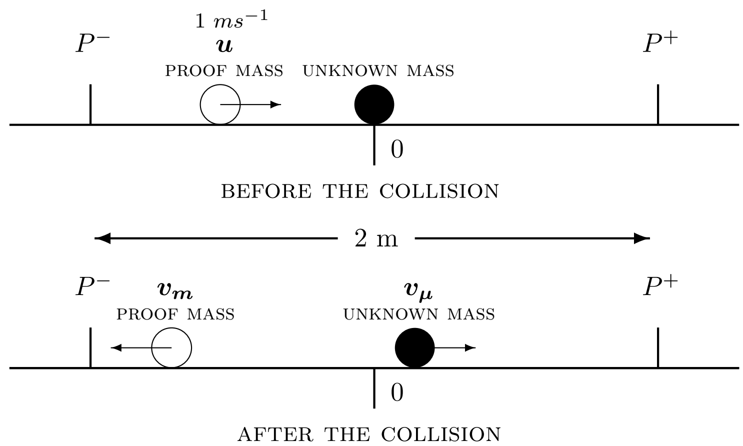

Fig. 1. COLLIDER MACHINE experiment.

$$\frac{1}{2}mu^2 = \frac{1}{2}mv_m^2 + \frac{1}{2}\mu v_\mu^2, \tag{6}$$

that can be solved for v_m and v_μ:

$$v_m = \frac{m-\mu}{m+\mu}\ u, \tag{7}$$

$$v_\mu = \frac{2m}{m+\mu}\ u. \tag{8}$$

From these formulae we see that after a collision:

(a) if $m < \mu$, then the proof particle move backwards after the collision;

(b) if $m > \mu$, then the proof particle will move forward; and

(c) if $m = \mu$, then the proof particle of mass m comes to rest and the particle of unknown mass μ is projected forward with the previous value of the speed of the proof particle.

This experiment can be designed to measure the unknown mass μ, using proof particles of known mass m projected at the same speed u.

We establish the convention that the particle of unknown mass is placed at the origin of coordinates and points $P^- \equiv -1\ m$ and $P^+ \equiv$

$+1\ m$ are the *flags* of the experimenter's observations: when the proof particle is seen crossing the points P^- or P^+ the experiment terminates. If the proof mass crosses the flag P^- then we have $m < \mu$ (as depicted in Figure 1), and if it crosses the flag P^+, we have $m > \mu$.

For this experiment there are various facts that are largely irrelevant, or where errors can be tolerated. These include the (finite) distance between the two flags, the precision of the placement of the flags, the error in placing the particle of the unknown mass at the origin (let us say approximately $0\ m$), and the initial speed of the proof particle (let us say approximately $1\ ms^{-1}$). Note that the observed velocities of the particles after the collision, after crossing one or both the flags, are irrelevant.

However quantities and facts that are relevant include: the one dimensional character; that the masses of the unknown particles are continuous variable in the range $(0, 1)$; that the particle of unknown mass μ is at rest; and that the collisions are elastic.

Looking closer to the experiment, we however find an experimental barrier: the time for the proof particle crossing the distance of $1\ m$ after the collision is given by

$$t_{exp} \;=\; \frac{1}{u}\left|\frac{m+\mu}{m-\mu}\right| \;, \tag{9}$$

that, for the values we will take of the masses and initial speed, is of the order of

$$\frac{A}{|m-\mu|} \;\leq\; t_{exp} \;\leq\; \frac{B}{|m-\mu|} \;, \tag{10}$$

for some constants A and B.

3.3 CME as Oracle

In the *shooting state* the machine prepares and fires a proof particle of mass m as detailed above. The experiment continues until the proof particle crosses one of the flags $P^{\pm}$, and then returns a state $m < \mu$ or $m > \mu$ to the Turing machine.

The Turing machine is connected to the collider experiment CME in the same way as it would be connected to an oracle: we replace the query state with a *shooting state* (q_s), the "yes" state with a *lesser state* (q_l), and the "no" state with a *greater state* (q_g). The resulting computational device is called the *(analogue-digital) collider machine experiment.*

In order to carry out an experiment, the machine will write a word z in the query tape and enter the shooting state. The word z codes for a dyadic rational mass m of the "proof" particle. In the *shooting state* the machine prepares and fires a proof particle of mass m as detailed above. The experiment continues until the proof particle crosses one of the flags $P^{\pm}$, and then returns a state $m < \mu$ or $m > \mu$ to the Turing machine.

Technically, this word z will either be "1", or a binary word beginning with 0. We will use y ambiguously to denote both a word $y_1 \ldots y_n \in \{1\} \cup \{0s : s \in \{0,1\}^*\}$ and the corresponding dyadic rational $\sum_{i=1}^{n} 2^{-i+1} y_i \in [0,1]$. In this case, we write $|y|$ to denote n, i.e., the size of $y_1 \ldots y_n$.

Consider the precision of the experiment. When measuring the output state the situation is simple: either the proof particle of mass m crosses P^- or it crosses P^+ (or, after some timeout, no proof particle is detected). Errors in observation do not arise. There are different postulates for the precision of the experiment, and we list some in order of decreasing strength:

Definition 5. *The CME is error free if the mass of proof particle can be set exactly to any given dyadic rational number. The CME is error prone with arbitrary precision if the mass of proof particle can be set only to within a non-zero, but arbitrarily small, dyadic precision. The CME is error prone with fixed precision if there is a value $\varepsilon > 0$ such that the mass of proof particle can be set only to within a given precision ε.*

3.4 Bisection algorithm

Now we can describe the algorithm in full detail. Let $T : \mathbb{N} \to \mathbb{N}$ be the time given for the experiment to take place as a function (total map) of the size of the sequence of bits setting the value of the mass of the proof particle. The function T can be seen as a *schedule*, i.e., in each experiment, in order to read the $|m|$-th bit of the mass μ, $T(|m|)$ gives *the amount of time steps that the experimenter is prepared to wait* until resuming the experimental conditions. The function T can either be a computable function or a non-computable function of its argument.

After setting the mass m, the CME will fire a proof particle of mass m, wait $T(|m|)$ time units, and then check if the particle crossed one of the flags. If the particle crossed the flag P^-, then the Turing machine computation will be resumed in the state q_l. If the particle crossed the flag P^+, then the Turing machine computation will be resumed in the state q_g. Perhaps, after time $T(|m|)$, no proof particle is detected.

Bisection(t) — THE BISECTION ALGORITHM: A PROCEDURE TO READ THE FIRST n BITS OF A UNKNOWN MASS μ

1. **input** n — required precision coded by the number of places to the right of the left leading 0;
2. $m_1 := 0$, $m_2 := 1$, $m := 0$ — initial values with no physical significance; note $|m_1| = 0$, $|m_2| = 1$, and $|m| = 0$;
3. **while** $|m| \leq n$ **do**
   (a) $m := \frac{m_1+m_2}{2}$;
   (b) place the particle of unknown mass $\mu \in [0, 1]$ at the origin;
   (c) project proof particle of mass m to collide with particle of unknown mass;
   (d) **if** proof particle crosses the flag P^- in time $T(|m|)$ **then** $m_1 := m$; append 1; — it is known that $\mu \in]m, m_2[$;
   (e) **if** proof particle crosses the flag P^+ in time $T(|m|)$ **then** $m_2 := m$; append 0; — it is known that $\mu \in]m_1, m[$;
   (f) **if** no particle crosses the flags in time $T(|m|)$ **then** return time out;
4. **end while**;
5. **output** dyadic rational denoted by m.

The bisection method applies to each type of precision.

3.5 Notions of measurable

Definition 6. *A mass μ is said to be* measurable *if there exists a schedule T such that the digits of μ can be computed by performing the collision experiment repeatedly. Otherwise, the mass is said to be* non-measurable*.*

Definition 7. *A mass μ is said to be* effectively measurable *if there exists a computable schedule T such that the digits of μ can be computed by performing the collision experiment repeatedly. Otherwise, the mass is said to be* effectively non-measurable*.*

To measure time we need to make step counting and time explicit inside the machine. To introduce a *system clock* as part of the Turing machine we can employ the concept of a *time constructible function*, introduced by Hartmanis in 1965.

Definition 8. *A total function $f : \mathbb{N} \to \mathbb{N}$ is said to be time constructible if there is a Turing machine $\mathcal{M}$ such that, for all $n \in \mathbb{N}$ and all inputs of size n, $\mathcal{M}$ halts in exactly $f(n)$ steps.*

Definition 9. *A mass μ is said to be* feasible *if there exists a time constructible computable schedule T such that the digits of μ can be computed by performing the collision experiment repeatedly. Otherwise, the mass is said to be* non-feasible*.*

3.6 Notions of computation

Definition 10. *An* error free analogue-digital collider machine *is a Turing machine connected to an error prone CME. In a similar way, we define an* error prone analogue-digital collider machine with arbitrary precision*, and an* error prone analogue-digital collider machine with fixed precision*.*

If an error prone analogue-digital collider machine, with unknown mass $\mu \in (0, 1)$, is triggered by the proof particle with dyadic rational mass $z \in [0, 1]$, then we are certain that the computation will be resumed in the state q_l if $m < \mu$, and that it will be resumed in the state q_g when $m > \mu$. We define the following decision criteria:

Definition 11. *Let $A \subseteq \Sigma^*$ be a set of words over Σ. We say that an error free analogue-digital collider machine $\mathcal{M}$* decides A *if there exists a time constructible schedule t to operate the coupled CME and an oracle μ such that, for every input $w \in \Sigma^*$, w is accepted if $w \in A$ and rejected when $w \notin A$. We say that $\mathcal{M}$* decides A in polynomial time*, if $\mathcal{M}$ decides A, and there is a polynomial p such that, for every $w \in \Sigma^*$, the number of steps of the computation is bounded by $p(|w|)$.*

Definition 12. *Let $A \subseteq \Sigma^*$ be a set of words over Σ. We say that an error prone analogue-digital collider machine $\mathcal{M}$* decides A *if there exists a time constructible schedule t to operate the coupled CME with a given oracle μ and a number $\gamma < \frac{1}{2}$, such that the error probability of $\mathcal{M}$ for any input w is smaller than γ. We call* correct *to those computations which correctly accept or reject the input. We say that $\mathcal{M}$* decides A in polynomial time*, if $\mathcal{M}$ decides A, and there is a polynomial p such that, for every input $w \in \Sigma^*$, the number of steps in every computation of $\mathcal{M}$ on w is bounded by $p(|w|)$.*

We can end this section with some results about questions that are experimentally undecidable:

Proposition 4. *That the proof mass coincides with the given unknown mass cannot be established experimentally in finite time by the CME.*

Proof: According to Equation 10, as $m \to \mu$ through the bisection method, the time the experimenter has to wait goes to infinity, $t_{exp} \to +\infty$. If the two masses coincide, then the experimenter will never know. □

As a trivial consequence of this statement we have the folowing theorem:

Proposition 5. *To know if the unknown mass is a dyadic rational cannot be established experimentally in finite time by the CME.*

And, finally, one important statement to keep in memory for the sections to follow, and its fundamental consequence.

Proposition 6. *At each stage of the bisection algorithm, the lower bounds on the time of a single experiment with the CME are exponential in the size of the mass of the proof particle.*

Proof: We know that the time taken by a single experiment is given by Equation 10 at step n with $|m| = n$. Thus μ has a pattern of the form $\mu = m \pm m' \times 2^{-n'-1}$, with $m' \in [0, 1]$ and $n' > n$, and t_{exp} has a pattern of the form

$$t_{exp} \sim \frac{K}{|m - (m \pm m' \times 2^{-n'-1})|} ,$$

that is,[9]

$$t_{exp} \sim \frac{K}{| \pm m' \times 2^{-n'-1}|} \in \Omega(2^n) .$$

□

Thus, we have the following consequence:

Proposition 7. *The protocol that processes queries between a Turing machine and the collider takes time that is at least exponential in the size of the mass of the proof particle specified by the queries.*

4 Geroch-Hartle on computability and measurement

Let us consider the reflections of physicists Geroch and Hartle on computability and measurement ([15]). Several of their speculations and questions are analysed formally in our theory.

Geroch and Hartle start by considering the concept of *measurable number* in contrast to the concept of *computable number*:

[9] Let f and g be total maps with signature $\mathbb{N} \to \mathbb{N}$. We say that $f \in \Omega(g)$ if there exists a constant $k \in \mathbb{R}$ such that, for an infinite number of values of $n \in \mathbb{N}$, $f(n) > kg(n)$.

Geroch-Hartle 1 We propose, in parallel with the notion of a computable number in mathematics, that of a measurable number in a physical theory. The question of whether there exists an algorithm for implementing a theory may then be formulated more precisely as the question of whether the measurable numbers of the theory are computable.

Then they add some considerations on numbers being measurable and/or computable:

Geroch-Hartle 2 We argue that the measurable numbers are in fact computable in the familiar theories of physics, but there is no reason why this need be the case in order that a theory have predictive power. Indeed, in some recent formulations of quantum gravity as a sum over histories, there are candidates for numbers that are measurable but not computable.

They introduce the notion of a technician measuring physical variables:

Geroch-Hartle 3 Regard number w as measurable if there exists a finite set of instructions for performing an experiment such that a technician, given an abundance of unprepared raw materials and an allowed error ε, is able by following those instructions to perform the experiment, yielding ultimately a rational number within ε of w.

The accuracy ε is to be understood as arbitrarily small. The technician and set of instructions, together with some memory to take account of intermediate calculations, we replace by a *Turing machine*. In our model of measurement embodied in Principle 6, the Turing machine *represents formally* the physicist or the experimenter. Thus, we propose the assumption:

Thesis 1 *The experimenter following his or her instructions is modelled by a Turing machine. The measuring process is controlled by an algorithm that runs on the machine, generating the atomic instructions, specified by theory $\mathcal{T}$, to be performed at each step of the experimental procedure.*

This postulate says that the experimenter cannot escape *the logic of following a set of rules* as formalised by computability theory; and, of course, that the logic of experimental procedures can be captured completely by a Turing machine.

A point not considered in [15] is that not all measurements are possible. Assuming the physicist to be a Turing machine, then the limits of Turing machine computation can determine limits on measurements and, therefore, on the nature of physical experiments.

As we will see in Section 6, our work makes the concept of *measurable* as precise as the concept of *computable*. Now this was *not* the intention of [15]:

Geroch-Hartle 4 "Measurable" is analogous to, although of course much less precise than, "computable". The technician is analogous to the computer, the instructions to the computer program, the "abundance of unprepared raw materials" to the infinite number of memory locations, initially blank. Indeed, one can think of the measurable numbers as those that are "computable" using an analog, rather than digital, computer.

Geroch and Hartle stress need for a theory to specify a *gedankenexperiment* as follows:

Geroch-Hartle 5 The notion "measurable" involves a mix of natural phenomena and the theory by which we describe those phenomena. Imagine that one had access to experiments in the physical world, but lacked any physical theory whatsoever. Then *no* number w could be shown to be measurable, for, to demonstrate experimentally that a given instruction set shows w measurable would require repeating the experiment an infinite number of times, for a succession of εs approaching zero. One could not even demonstrate that a given instruction set shows measurability of any number at all, for it could turn out that, as ε is made smaller, the resulting sequence of experimentally determined rationals simply fails to converge. It is only a *theory* that can guarantee otherwise. The situation is analogous to that of trying to demonstrate that a given Fortran program shows some number to be computable. There is no general algorithm for deciding this. In particular, it would not do merely to run the program for a few selected values of ε.

Now, how does the Turing machine communicate with Nature? We believe that this interaction is captured by the concept of the continuing evolution of a physical experiment acting as an oracle.

Thesis 2 *The measurement apparatus is taken to be an oracle to a Turing machine. The interaction is achieved through a protocol which counts time. After each consultation, the oracle may provide one bit of the measurement. This bit also provides the necessary information to the machine to proceed with the experimental procedure.*

Geroch and Hartle argue that *every computable number is measurable.* A few paragraphs further on, Geroch and Hartle provide the flavour of a *proof.* This proof is given to the reader by the following:

Geroch-Hartle 6 This is easy to see: Let the instructions direct that the raw materials be assembled into a computer, and that a certain Fortran program — one specified in the instructions — be run on that computer. That is, every digital computer is at heart an analog computer.

Then the authors ask the following question:

Geroch-Hartle 7 We now ask whether, conversely, every measurable number is computable — or, in more detail, whether current physical theories are such that their measurable numbers are computable. This question must be asked with care.

Actually, the question received a very careful answer in our [9]: *the experiment SME demonstrates that there are numbers that are measurable in Newtonian dynamics but that are not computable.*

5 The laws of dynamics

In this section we explain how the collider experiment lies at the heart of measuring masses in Classical Mechanics. Our aim is to define formally the measurement function for (inertial) mass from Newtonian dynamics.

First law. The first law of Newton establishes that *a particle not subjected to a net force will move in a uniform motion in a straight line.* Since the motion of a particle has to be specified with respect to a particular reference frame, the content of the first law can only be understood if such a reference frame is provided. Also, looking at the statement of the first law, we see that the concept of *force* was not yet defined. The first law should be regarded in the following way: in a region of space containing the particle, far away from all other matter, we can always define a reference frame with respect to which that particle will move in a uniform motion in a straight line. Such a reference frame is the *inertial reference frame*; an example is that of the stars — Kepler's reference frame[10].

Second law. Having found a inertial reference frame, the departure from a uniform motion in a straight line is "measured" by the kinematic concept of acceleration. The departure from a constant speed in a straight line should be due to a *force* that is impressed on the particle by some physical process. If $\boldsymbol{v}$ is the velocity of a particle in that reference frame, in an arbitrary instant of time t, its acceleration $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$ will be non-zero, and this quantity will be a convenient measure of the force $\boldsymbol{f}$ being applied.

In accordance with the Aristotelian principle that causes should be proportional to their effects, Newton assumed that $\boldsymbol{f}$ is proportional to $\boldsymbol{a}$, or $\boldsymbol{f} = m\boldsymbol{a}$, where m is the coefficient that will depend on the particle

[10] The reference frame of the stars is a good inertial frame for experiments carried out on Earth.

under consideration and that we will call *(inertial) mass*.[11]

Third law. According to Newton's third law, when two particles P and Q interact, the force applied on P by virtue of Q is equal to the force applied on Q by virtue of P, but of opposite direction.

Newton defined *momentum* $\boldsymbol{p}$ of a particle as the product of its inertial mass m by its velocity $\boldsymbol{v}$.[12] Taken together, the second and the third laws give rise to the law of *conservation of momentum* that implies that the sum of momenta of two particles before a collision is equal to the sum of momenta of the same particles after that collision. If μ and 1 are the masses of the two particles a and b, respectively, and $\boldsymbol{u}_a$ and $\boldsymbol{0}$ are their respective velocities immediately before the collision, and $\boldsymbol{v}_a$ and $\boldsymbol{v}_1$ are their velocities immediately after the collision, then

$$\mu \boldsymbol{u}_a = \mu \boldsymbol{v}_a + \boldsymbol{v}_1 \tag{11}$$

that is

$$\mu = \frac{\|\boldsymbol{v}_1\|}{\|\boldsymbol{u}_a - \boldsymbol{v}_a\|} \tag{12}$$

and

$$(\boldsymbol{u}_a - \boldsymbol{v}_a)\mu = \boldsymbol{v}_1 \ . \tag{13}$$

This last equation implies that the vectors $\boldsymbol{u}_a - \boldsymbol{v}_a$ and $\boldsymbol{v}_1$ are colinear, a result that constitutes the essence of the third law of Newton. For the unidimensional collider, Equation 12 can be rewritten with the velocity scalars:

$$\mu = \frac{v_1}{u_a - v_a} \tag{14}$$

where u_a and v_1 are always positive and v_a, speed of the particle of proof mass, can be either negative or positive depending on its behaviour after the collision — bouncing back or going forward.

The determination of mass. These equations show that the third law is also the way to ascertain the value of the coefficient called *mass*. Equation 12 gives the mass of an arbitrary particle using a standard particle

[11] To Aristotle the *force* applied is the cause and in some way the velocity is the *effect*. Since uniform motion in a straight line does not need any explanation, Newton searched for the variation of uniform motion in a straight line as the required effect.

[12] In the *Principia*, Newton defined force as change of momentum, i.e., $\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt}$.

(of mass 1 Kg): this value can be measured in a collision experiment. Thus, if one of the particles is chosen as unit, then the masses of all other particles can be determined by making them collide with the standard particle. Consider a possible measurement map M for mass.

The *inertial mass* $M(a)$ of a particle a, as determined by the collider and velocity measurements only, is defined by Equation 14 rewritten in the form:

$$M(a) = \frac{v_1}{u_a - v_a} \ , \tag{15}$$

where $\boldsymbol{u}_a$ and $\boldsymbol{v}_a$ are the velocities of particle a before and after the collision, and $\boldsymbol{v}_1$ is the velocity after the collision of the standard reference particle. Here are some simple consistency theorems:

Proposition 8. *$M(a) < M(b)$ if, and only if, the particle a of mass μ bounces back when projected towards the particle b of mass μ' at rest.*

Proof: By Equation 7, we have that

$$v_a = \frac{\mu - \mu'}{\mu + \mu'} \ u_a \ ,$$

where the sign of v_a is decided by the difference $\mu - \mu'$. Thus, we only have to prove that $\mu < \mu'$. But, since $M(a) < M(b)$, we conclude

$$\frac{v_1}{u_a - v_a} < \frac{v'_1}{u_b - v_b} \ ,$$

if, and only if,

$$\frac{\mu v_1}{\mu u_a - \mu v_a} < \frac{\mu' v'_1}{\mu' u_b - \mu' v_b} \ ,$$

and, by conservation of momentum, if, and only if,

$$\frac{\mu v_1}{v_1} < \frac{\mu' v'_1}{v'_1} \ ,$$

and, finally, if, and only if, $\mu < \mu'$. $\square$

In a similar way, it is straighforward to prove that:

Proposition 9. *$M(a) = M(b)$ if, and only if, the particle a of mass μ becomes at rest when projected towards the particle b with the same mass at rest.*

The basic question is: Does the CME implement a *comparative concept* supporting a *formal measurement M* in the sense of Hempel? Does M qualify as a measurement function? We will see that, indeed, we have both a comparative concept and a measurement.

6 Refinement of the theory of measurement

6.1 Measuring quantities

Suppose that we wish to measure an attribute of an object of $\mathcal{O}$ using real numbers. We need a map $M : \mathcal{O} \to \mathbb{R}$ assigning to each object $a \in \mathcal{O}$ an attribute value $M(a)$. Such a map cannot be chosen arbitrarily. To qualify as a measurement in an empirical science, an experiment must be conceived that "validates" or "witnesses" the definition. The experimental apparatus works with the objects from $\mathcal{O}$, allowing the experimenter to compare different objects with respect to a given attribute. The outcome of each experiment is an event that tells us whether or not the attribute of object a is less than the attribute value of object b. Observing the equipment, there will be an event for "yes", an event for "no", and an event for "don't know". As we will see shortly, in our theory, "don't know" is an event "experiment timed out". With time in mind, we adapt the notation in Section 2.2: in the bipolar subset of events we replace 0 with $\perp$ ("undefined") to mark that the binary equivalence $\mathcal{E}$ is true.

Let us assume there is a time $t \in \mathbb{N}$ associated to each experiment. A collection of such times constitute the schedule of the collider protocol. In all measurement procedures in this paper, the experimenter — the Turing machine — generates a possibly infinite sequence of binary words $\{z_i\}_{i\in\mathbb{N}}$. If the time schedule of oracle consultation allows, then this sequence converges into the unknown real ζ being measured (in its binary expansion).

For the purpose of what follows, every number ζ can be seen as an infinite binary string. We don't accept infinite suffixes of 1s to denote dyadic rationals. If a sequence is finite, then we consider an infinite number of 0s padded to its right. The concept of limit induces a topology over the set of finite and infinite binary sequences $\{0, 1\}^\omega$.

Definition 13. *We say that the sequence of binary words $\{z_i\}_{i\in\mathbb{N}}$ converges to ζ if (a) for all $i \in \mathbb{N}$, z_i is a finite sequence, (b) for all $i \in \mathbb{N}$, z_i is a prefix of ζ, and (c) for each prefix z of ζ, there is a $i \in \mathbb{N}$ such that z is a prefix of z_i.*

Each experimental apparatus $\mathcal{A}$ we have explored so far is specified by a physical theory $\mathcal{T}$ and is designed to measure a real number ζ. Let $\mathcal{A}(\mathcal{T}, \zeta)$ denote the experimental apparatus together with the quantity. We are able to define precisely the notion of a measurable number:[13]

[13] Compare the context of [15] and [1, 5, 3].

Definition 14. *Let $\mathcal{A}(\mathcal{T}, \zeta)$ be an experimental apparatus for physical theory $\mathcal{T}$ and physical quantity ζ. The number ζ is* measurable *if the Turing machine equipped with the physical oracle $\mathcal{O}(\mathcal{T}, \zeta)$ and a time schedule can produce an infinite sequence of prefixes of ζ, $\{z_i\}_{i \in \mathbb{N}}$, without timing out in any query, such that*

$$\lim_{i \to \infty} z_i \ = \ \zeta \ . \tag{16}$$

In the bisection method, the infinite sequence of queries is almost such a sequence $\{z_i\}_{i \in \mathbb{N}}$, but *not quite* since each query may differ in the last bit from a prefix of the unknown number being measured. We define the meet operation, which allows us to identify the largest common prefix to two given words over the same alphabet Σ:

Definition 15. *Let α and β be two finite or infinite words over the same alphabet Σ. We define the meet $\alpha \sqcap \beta$ as the finite word γ over Σ, if it exists, such that (a) γ is prefix of both α and β and (b) if δ over Σ is prefix of both α and β, then δ is a prefix of γ. It such a prefix does not exist we say that the meet is undefined.*

Thus, according with our previous analysis of experimental situations, the sequence of queries involved in the bisection procedure has the following property: if ζ is measurable, then the sequence $\{z_i \sqcap \zeta\}_{i \in \mathbb{N}}$ converges to ζ. Notice that, whenever one of the words over Σ is finite, the meet is always defined. If the meet is undefined, we say that its size is infinite. The following proposition is straightforward to prove:

Proposition 10. *Let $\mathcal{A}(\mathcal{T}, \zeta)$ be an experimental apparatus for physical theory $\mathcal{T}$ and physical quantity ζ. The number ζ is* measurable *if, and only if, a Turing machine with physical oracle $\mathcal{O}(\mathcal{T}, \zeta)$ can produce an infinite sequence of queries $\{z_i\}_{i \in \mathbb{N}}$ such that*

$$\lim_{i \to \infty} z_i \sqcap \zeta \ = \ \zeta \ . \tag{17}$$

6.2 Measurement axioms with time

We begin with some properties of abstract binary relations indexed by a real parameter "time" $t > 0$ on a set $\mathcal{O}$.

Definition 16. *A relation $\mathcal{E}_t$ in $\mathcal{O} \times \mathcal{O}$, for the time bound $t > 0$, is said to be a* timed equivalence relation *if there is a $K \geq 1$ so that*
(a) $\mathcal{E}_t$ is reflexive,

(b) $\mathcal{E}_t$ is timed symmetric*: for every a, b in $\mathcal{O}$, if $a\mathcal{E}_t b$, then $b\mathcal{E}_{t/K} a$,*
(c) $\mathcal{E}_t$ is timed transitive*: for every a, b,and c in $\mathcal{O}$, if $a\mathcal{E}_t b$ and $b\mathcal{E}_t c$, then $a\mathcal{E}_{t/K} c$,*
(d) if $t < t'$, then $a\mathcal{E}_{t'} b \;\Rightarrow\; a\mathcal{E}_t b$.

Definition 17. *Two binary relations $\mathcal{E}_t$ and $\mathcal{L}_t$ ($t > 0$) determine a* timed comparative concept *for the elements of $\mathcal{O}$, if*
(a) $\mathcal{E}_t$ is a timed equivalence relation,
(b) there is a $K \geq 1$ so that for every a, b, c in $\mathcal{O}$, if $a\mathcal{L}_t b$ and $b\mathcal{L}_t c$, then $a\mathcal{E}_{t/K} c$,
(c) for all $t > 0$ and $a, b \in \mathcal{O}$, exactly one of $a\mathcal{E}_t b$, $a\mathcal{L}_t b$, $b\mathcal{L}_t a$ holds,
(d) if $t < t'$, then $a\mathcal{L}_t b \;\Rightarrow\; a\mathcal{L}_{t'} b$.

Note that Definition 17(c) summarises the ideas of *irreflexivity* and *connectedness.*

Note also that, although property 16(d) is kept explicitly, it can be omitted, since it is derivable from the other properties listed in Definition 16 and those listed in Definition 17.

Proposition 11. *If $t < t'$, then $a\mathcal{E}_{t'} b \;\Rightarrow\; a\mathcal{E}_t b$.*

Proof: Suppose that $a\mathcal{E}_{t'} b$ holds. Then $a\mathcal{L}_{t'} b$ does not hold, due to property Definition 17(c). We conclude, by Definition 17(d), that $a\mathcal{L}_t b$ does not hold. Then, either $b\mathcal{L}_t a$ or $a\mathcal{E}_t b$ holds. If $b\mathcal{L}_t a$ holds, then $b\mathcal{L}_{t'} a$ holds and $a\mathcal{E}_{t'} b$ cannot hold, by Definition 17(c), which is against the hypothesis. Thus $a\mathcal{E}_t b$ is the case. $\square$

Now suppose we have an experimental apparatus for making measurements. This takes the form of some form of comparison of two objects in $\mathcal{O}$ taking place in a given time $t > 0$. (The time t is allowed to vary over real values for convenience, but there would be no problem in restricting it to rational values, or with slight modification to some formulae, integer values.) The possible outcomes for the experiment are labelled $\{-1, \perp, +1\}$, where $\perp$ should be thought of as "no answer". We will now define, for all $t > 0$, binary relations $\mathcal{E}_t$ and $\mathcal{L}_t$ on $\mathcal{O}$ by using this experiment. Later we shall discuss when these relations obey Definition 17.

Definition 18. *Whenever the experiment is done with arbitrary objects $a, b \in \mathcal{O}$, if the outcome in time t is event -1, then $a\mathcal{L}_t b$ is the case, if the outcome in time t is event is $+1$, then $b\mathcal{L}_t a$ is the case, and if the outcome in time t is "no answer" ($\perp$), then $a\mathcal{E}_t b$ is the case.*

Definition 19. *Let $\mathcal{E}_t$ and $\mathcal{L}_t$ be timed comparative relations on the set $\mathcal{O}$ of objects (Definition 17). Suppose there exists an* experimental apparatus *to witness these relations, as in Definition 18. Then the map* $M : \mathcal{O} \to \mathbb{R}$ *is a* measurement map *if*

1. *For all time* $t > 0$*, if* $a\mathcal{L}_t b$ *holds, then* $M(a) < M(b)$*.*

Considering the real $M(a)$, for the object $a \in \mathcal{O}$, as an infinite binary sequence, we denote by $M(a)\restriction_n$ the dyadic rational corresponding to the prefix of size n of $M(a)$ and by a_n an object from $\mathcal{O}$ with that measure. Such an object a_n exists due to the convention of the toolbox of standards: once specified the *unit*, we have access to all its multiples and submultiples.

Definition 20. *The* complexity of a measurement map $M : \mathcal{O} \to \mathbb{R}$*, given the timed comparative relations* $\mathcal{E}_t$ *and* $\mathcal{L}_t$ *on the set* $\mathcal{O}$ *of objects, is the map* $T : \mathbb{N} \to \mathbb{N}$ *defined as follows:*

$$T(n) = \min\{t \in \mathbb{N}\setminus\{0\} : a_n\mathcal{L}_t a \text{ for some } a, a_n \in \mathcal{O} \text{ with } M(a_n) = M(a)\restriction_n\}.$$

For the *collider machine experiment*, the complexity of the measurement map is *exponential*. This complexity of measurement is, indeed, a lower bound on the time needed to get an answer from the machine, as can be seen in the proof of Proposition 6.

Now, we introduce an extra axiom for the physical apparatus:

Definition 21. *The apparatus satisfies the* separation property *for the measurement map* $M : \mathcal{O} \to \mathbb{R}$ *if for every objects* a *and* b *in* $\mathcal{O}$*, if* $M(a) < M(b)$*, then there exists a time bound* t *such that* $a\mathcal{L}_t b$*.*

To connect these ideas with Hempel's axiomatisation, we use the following definition:

Definition 22. *Given the timed comparative concept* $\mathcal{E}_t$ *and* $\mathcal{L}_t$*, for some time bound* t*, we define the following relations* $\mathcal{E}_{lim}$ *and* $\mathcal{L}_{lim}$*:*
(a) for every a *and* b *in* $\mathcal{O}$*,* $a\mathcal{E}_{lim}b$ *if* $a\mathcal{E}_t b$ *for every time bound* t*, and*
(b) for every a *and* b *in* $\mathcal{O}$*,* $a\mathcal{L}_{lim}b$ *if there exists a time bound* t *such that* $a\mathcal{L}_t b$*.*

Proposition 12. *If the two relations* $\mathcal{E}_t$ *and* $\mathcal{L}_t$ *define a timed comparative concept (Definition 17) and the physical apparatus witnessing the relations satisfies the separation property (Definition 21), then the two relations* $\mathcal{E}_{lim}$ *and* $\mathcal{L}_{lim}$ *define a comparative concept and* M *is a measurement map in the sense of Hempel (see Definitions 3 and 4).*

Proof: We have to prove that Hempel's axiomatization holds, which is straightforward.

(1) $\mathcal{E}_{lim}$ is reflexive: Suppose that, for some object a in $\mathcal{O}$, $a\mathcal{E}_{lim}a$ does not hold. It means that, for some time bound t, $a\mathcal{E}_t a$ does not hold, which is a contradiction with the fact that $\mathcal{E}_t$ is reflexive.

(2) $\mathcal{E}_{lim}$ is symmetric: Use Definition 16(b).

(3) $\mathcal{E}_{lim}$ is transitive: Use Definition 16(c).

(4) $\mathcal{L}_{lim}$ is transitive: Use Definition 17(b).

(5) $\mathcal{L}_{lim}$ is $\mathcal{E}_{lim}$-irreflexive: Suppose that, for some objects a and b in $\mathcal{O}$, both $a\mathcal{E}_{lim}b$ and $a\mathcal{L}_{lim}b$ hold. Then, there is a time bound t such that $a\mathcal{L}_t b$. Since $\mathcal{L}_t$ is $\mathcal{E}_t$-irreflexive, we conclude that $a\mathcal{E}_t b$ does not hold, which is contradictory with the case that $a\mathcal{E}_{lim}b$ holds.

(6) $\mathcal{L}_{lim}$ is $\mathcal{E}_{lim}$-connected: Suppose that, for some objects a and b in $\mathcal{O}$, $a\mathcal{E}_{lim}b$ does not hold. Then, there is a time bound t such that $a\mathcal{E}_t b$ does not hold. Consequently, since $\mathcal{L}_t$ is $\mathcal{E}_t$-connected, either $a\mathcal{L}_t b$ or $b\mathcal{L}_t a$, meaning that either $a\mathcal{L}_{lim}b$ or $b\mathcal{L}_{lim}a$.

(7) Suppose that $M(a) \neq M(b)$. Then either $M(a) < M(b)$ or $M(a) > M(b)$. Consider the first case. By the separation property (Definition 21), there exists a time bound t such that $a\mathcal{L}_t b$ holds. Consequently, $a\mathcal{E}_t b$ is not the case and, therefore, $a\mathcal{E}_{lim}b$ is not the case.

(8) If $a\mathcal{L}_{lim}b$, then there exists a time bound t such that $a\mathcal{L}_t b$ and, consequently, $M(a) < M(b)$.

And we are done! □

6.3 The collider as an example

Now we are in a position to prove that the CME is a measuring process and that the mass obtained by the collision experiment is a measurement map. We use $\mathcal{O}$ to denote the set of objects used in the collider experiment. For the collider experiment, we measure mass using Equation 15, which is independent of the value of the initial velocity. The vital fact to remember is that the time t_{exp} taken to conclude the physical experiment for masses m_a and m_b is bounded by (for constants $A, B > 0$):

$$\frac{A}{|m_a - m_b|} \ \leq \ t_{exp} \ \leq \ \frac{B}{|m_a - m_b|} \ . \tag{18}$$

Proposition 13. *The map M, given values by Equation 15, is a measurement map with exponential complexity. That is, the collider provides a model of the timed axioms of measurement.*

Proof: We start by providing the semantics of the predicates $\mathcal{E}_t$ and $\mathcal{L}_t$. We say that two objects a and b have *experimentally* the same mass — event $\perp$ — if when a collides with b, there is no answer from the oracle in time t. We say that the object a has less mass than b if when a collides with b, the object a bounces back in time t.

Note that the separation axiom provided in Definition 21 is valid for the collider machine experiment: for every objects a and b in $\mathcal{O}$, if $M(a) < M(b)$, that is if $m_a < m_b$, then the time needed to detect the bouncing of object a is

$$t \;=\; \frac{B}{|m_a - m_b|}\ ,$$

that is, $a\mathcal{L}_t b$.

The $\mathcal{E}$-irreflexivity and $\mathcal{E}$-connectivity follow directly from the fact that the experimental outcomes (for a given setup) are exactly one of $\{-1, \perp, +1\}$. The properties 16(d) and 17(d) on increasing time are true, as a result of ± 1 at time t guarantees the same result for any time $t' > t$.

Let us prove that the predicate $\mathcal{E}_t$ is a timed equivalence relation.

It is *reflexive*: if two copies of a are made to collide, then there is no answer from the oracle at any time — event $\perp$. Consequently there will be no answer in time t.

It is *timed symmetric*: if a collides with b with no answer from the oracle in time t, then

$$\frac{B}{|m_a - m_b|} \;>\; t\ .$$

Then, if b collides with a, then

$$\frac{A}{|m_b - m_a|} \;> \frac{A}{B}t\ .$$

Thus, $a\mathcal{E}_t b \;\Rightarrow\; b\mathcal{E}_{A/B\,t}a$.

It is *timed transitive*: Suppose that a collides with b with no answer in time t, and that b collides with c with no answer in time t. Then

$$\frac{B}{|m_a - m_b|} \;>\; t \quad \text{and} \quad \frac{B}{|m_b - m_c|} \;>\; t\ .$$

Since

$$|m_a - m_c| \;=\; |m_a - m_b + m_b - m_c| \;\leq\; |m_a - m_b| + |m_b - m_c| \;,$$

we have

$$|m_a - m_c| \;<\; \frac{2\,B}{t}\;.$$

Now if a collides with c, there will be no answer in time $A/(2\,B)\,t$.

The proof that the predicate $\mathcal{L}_t$ is a transitive relation follows the same guidelines as the proof given immediately above. If a collides with b and bounces back in time t and b collides with c and bounces back in time t, then

$$\frac{A}{|m_a - m_b|} \;\leq\; t \quad \text{and} \quad \frac{A}{|m_b - m_c|} \;\leq\; t\;.$$

Since, in this case,

$$|m_a - m_c| \;=\; |m_a - m_b + m_b - m_c| \;=\; |m_a - m_b| + |m_b - m_c| \;,$$

the upper bound on the experimental time required to distinguish a and c is

$$\frac{B}{|m_a - m_c|} \;=\; \frac{B}{|m_a - m_b| + |m_b - m_c|} \;\leq\; \frac{B}{2\,A}\,t\;.$$

The complexity of the map is determined by the analysis done in the proof of Proposition 6. □

The theory of the *collider machine experiment* CME as a measurement device can be developped and fully axiomatized. Of course Hempel's timed system of axioms is not complete for the CME: many further complex properties of the CME can be axiomatised. Mainly, those properties that dissect the entanglement of the relations $\mathcal{E}_t$ and $\mathcal{L}_t$ for arbitrary values of t.

Let us give an example. In Hempel's system, it can be proved that, for every objects a, b, and c in $\mathcal{O}$, if $a\mathcal{L}b$ and $b\mathcal{E}c$, then $a\mathcal{L}c$. In the timed system, it does not hold that, for every objects a, b, and c in $\mathcal{O}$, if $a\mathcal{L}_t b$ and $b\mathcal{E}_t c$, then $a\mathcal{L}_t c$. But for the collider this theorem can be replaced by a timed one in the following form:

Proposition 14. *For every objects a, b, and c in $\mathcal{O}$, for every time bound t, there is a $K \geq 2$ so that the following holds: If $a\mathcal{L}_t b$ and $b\mathcal{E}_{Kt} c$, then $a\mathcal{L}_{Kt} c$.*

Proof: If $a\mathcal{L}_t b$ and $b\mathcal{E}_{t'} c$, then

$$t > \frac{A}{m_b - m_a} \quad \text{and} \quad t' < \frac{B}{|m_b - m_c|} \ .$$

If $t' \ = \ 2B/A \ t$ then we have

$$|m_b - m_c| \ < \ (m_b - m_a)/2 \ ,$$

and then

$$m_c - m_a \ \geq \ m_b - m_a - |m_b - m_c| \ > \ (m_b - m_a)/2 \ .$$

Then an upper bound on the time taken to distinguish a and c is

$$\frac{B}{m_c - m_a} \ < \ \frac{2\,B}{m_b - m_a} \ < \ \frac{2\,B}{A}\,t \ .$$

$\square$

Many propositions of this kind can be proved for the CME, namely introducing quantifiers. They show how masses can be compared in the less abstract timed system, where measurements take time, without further measurements.

We can also see how the CME fails to measure with arbitrary accuracy when used with a polynomial time limit:

Proposition 15. *Let $p(n)$ be a polynomial. For any a, a_n in $\mathcal{O}$ ($n \in \mathbb{N}$), such that $M(a_n) = M(a){\restriction}_n$, there are only finitely many n so that $a_n \mathcal{L}_{p(n)} a$.*

6.4 Complexity

We propose that a measurement procedure has a "computational complexity" that can be derived from the intrinsic duration of the phenomenon considered.

If a is the object being measured and, for all $i \in \mathbb{N}$, a_i is the object from the toolbox of standards corresponding to the dyadic rational z_i, then we can restate Proposition 10 in the following terms:

Proposition 16. *Let $\mathcal{A}(\mathcal{T}, \zeta)$ be an experimental apparatus for physical theory $\mathcal{T}$ and physical concept value ζ. If the Turing machine with the physical oracle $\mathcal{O}(\mathcal{T}, \zeta)$ and a schedule can give instructions to set an infinite sequence of objects $\{a_i\}_{i \in \mathbb{N}}$ to be compared with object a in some attribute, by the bisection method, without timing out in any query, then*

$$M(a) = \lim_{i \to \infty} M(a_i) \ . \tag{19}$$

Proposition 17. *If the Turing machine (experimenter) is equipped with the bisection algorithm, then the analogue-digital collider machine can serve as measurement apparatus for the measure of mass with complexity exponencial in the size of the query.*

Proof: The time of the experiment is exponential in the size of $z_i \sqcap \zeta$, where z_i is the i-th query and ζ the unknown mass. Using the bisection algorithm the size of the largest common prefix is $|z_i|$ up to 1 unit. Consequently, the time computed in this way is the same complexity class $(k' 2^{kn})$. $\square$

This last proposition shows that the bisection method is one of those methods that allows the experimenter, equipped with the toolbox of standards, to measure the unknown mass with a time schedule that does not depend on the unknown mass, although the experiment may time out assigning the two objects in the measurement context *the same mass* in the sense of relation $\mathcal{E}$.

We think these last propositions give a solid ground to understand our physical experiences of measurement and the role of the Turing machine as experimenter.

Now we introduce what we think is the most relevant concept:

Definition 23. *We say that a measurement in physical theory $\mathcal{T}$ has structural complexity T if the associated measurement map M has a computable complexity T in the sense of Definition 20.*

Then we can define complexity classes of measurements, such as:

Definition 24. *$\mathcal{T}-EXP$ is the class of measurements in physical theory $\mathcal{T}$ that have associated measurement maps with exponential time complexity, i.e., complexity $2^{O(n)}$.*

We can specify an open problem in measurement theory:

Conjecture 1. No reasonable physical measurement, based upon a reasonable physical theory $\mathcal{T}$, has an associated measurement map with polynomial time complexity.

The SME in [9] can be considered to be "unreasonable" since its behaviour is not *fully* governed by physical laws. This is because no physical law determines what happens in the "close vicinity" of the vertex of the wedge (cf. [14]).

7 The non-measurable character of a physical concept

We start with a definition more general than Definition 14:

Definition 25. *A number ζ is said to be* measurable *over a physical theory $\mathcal{T}$ if there exists a Turing machine M with experimental apparatus $\mathcal{A}(\mathcal{T}, \zeta)$, specified by the physical theory $\mathcal{T}$, and physical oracle $\mathcal{O}(\mathcal{T}, \zeta)$ which, running over unbounded time, computes a sequence of rational approximations to (the binary expansion of) ζ.*

(Compare the quotations Geroch-Hartle 3 and 5.) We are now going to reconsider the collider experiment in Section 3. Let ζ denote the unknown value to be measured and $\{z_i\}_{i\in\mathbb{N}}$ be the sequence of words queried by the Turing machine.

From the sequence $\{z_i \sqcap \zeta\}_{i\in\mathbb{N}}$, introduced in Section 6, we can extract the sequence of sizes $\{|z_i \sqcap \zeta|\}_{i\in\mathbb{N}}$, which determines the lower bound of the time needed to perform the i-th consultation of the experiment, $i \in \mathbb{N}$.

We suppose there is a notion of *physical time* that belongs to the physical theory $\mathcal{T}$ underlying the measurement. Suppose the natural physical $\mathcal{T}$-time of the experiment has a lower bound exponential in the size of the largest common prefix of the unknown word and the query word. Then the sequence of lower bounds in the times needed for the consultations is $\{2^{|z_i \sqcap \zeta|}\}_{i\in\mathbb{N}}$. Therefore, even if the program for the Turing machine "cheats" for some $i \in \mathbb{N}$, by timing out some queries, an infinite subsequence of queries has to have time constraints. The proper way to formulate this property is via the Ω notation:

Proposition 18. *Let $\mathcal{O}(\mathcal{T}, \zeta)$ be an oracle to a Turing machine for a physical theory $\mathcal{T}$ and physical quantity ζ. Let physical $\mathcal{T}$-time be τ. Let the oracle consultation schedule be T. If the number ζ is measurable then $T \in \Omega(\tau)$.*

Now, we make a conjecture, which we will call the BCT Conjecture, stating:

Conjecture 2. For all reasonable physical theories $\mathcal{T}$, for all reasonable physical measurements of ζ based upon $\mathcal{T}$, the natural physical $\mathcal{T}$-time τ is at least exponential in the size of $z \sqcap \zeta$, where z is a query of the experimenter.

Our Conjecture 2 claiming *exponential in the size of the query* can be explored for the bisection algorithm. By *exponential* we generally mean a law of time of the form

$$\tau(n) \;=\; 2^{kn} \;, \tag{20}$$

for some value of k different from 0.

As an example, consider the speed of light of 299 792 458 ms^{-1}. Any attempt to prove that it is 299 792 458. 0^{ω} ms^{-1} will fail, according to our conjecture, but an attempt to prove that it is 299 792 458. $0^{i}d$ ms^{-1}, for some large i may succeed for some digit $d \neq 0$.

Conjecture 2 is suggested by our studies of *gedankenexperimente* in a variety of physical fields, measuring *length*, *mass*, *resistance*, *latitude*, *mass of a elementary particle*, and *Brewster's angle* in optics. All these experiments are fully described in [7]. The conclusion of each analysis is the same: the time needed to establish the n-th bit of a value is at least exponential in n. Of course, if the statement of the conjecture is turned into a widely accepted thesis, or even a law about the process of measurement, then there will be deep consequences, both philosophical and physical.

The following propositions answer questions seen earlier in Section 4:

Proposition 19. *There are measurable numbers that are not computable.*

These are best seen through particular experiments such as [9].

Proposition 20. *There are computable numbers which are not measurable.*

Proof: Take any dyadic quantity ξ of size n and consider it measurable. Then, the Turing machine can produce a sequence $\{z_i\}_{i\in\mathbb{N}}$ of queries such that $\lim_{i\to+\infty} z_i = \xi$. As a consequence of the concept of limit provided by Definition 13, we know that there is an order $p \in \mathbb{N}$ such that, for $i > p$, $z_i = \xi$. For such queries z_i, $i > p$, the time of the experiment is infinite. □

This last Proposition 20 conspicuously challenges arguments in the quotation Geroch and Hartle 6 (recall Section 4). A reason is this: for Geroch and Hartle, a computable number is *a priori*, i.e., knowing that a number is computable we can prove it is computable. But, in our case, we do not know if a quantity being measured is computable or not.

We conclude that the Geroch and Hartle's Quotation 6 (see [15]) is a difficult one. Our interpretation is that Geroch and Hartle are making distinguishing those numbers which can *a priori* be known to be computable and, consequently, measurable, and those numbers under the influence of an experimental apparatus. Indeed, what Geroch and Hartle state in Quotations 5 and 6, taken together, is that *all computable numbers predicted by physical theories are measurable.* This view is acceptable when only negative results are in context. But for the Philosophy of Physics, if it is a refutation what we are looking for, then even this exercise of Geroch and Hartle is not suitable.

The diference of knowing and not knowing in advance if a given quantity is computable or not is entangled in the following two propositions from [4]. The first tells us that, if we know a quantity in advance, then we can design a schedule (using that quantity as a *conventional oracle* (!)) that allows the experimenter to measure the number:

Proposition 21. *There are programs N_k (with integer $k \geq 1$), with specified waiting times (say T_k), so that the following is true: For any non-dyadic $\mu \in [0, 1]$ and any $n \geq 0$, there is a k so that program N_k will find the first n binary places of μ.*

But if that quantity is not known in advance than, for most numbers, there is a last bit that can be read. (cf. Proposition 19, stated in advance for the purpose of clarity.)

Proposition 22. *There are uncountably many $\zeta \in [0, 1]$ so that, for any program P with a specified computable schedule, having access to the oracle $\mathcal{O}(\mathcal{T}, \zeta)$), there is an n so that P cannot determine the first n binary places of ζ.*

We note that the *impression* that the non-algorithmic character of measurement is induced by the thresholds of sensitivity of the equipment is false. In the collider machine experiment the two flags are put at a finite non-zero distance from each other: notice that the non-measurability arises no matter how small is the distance between the two flags. Besides that fact, there are uncomputable reals that are indeed measurable irrespective to the finite distance between flags of the collider.

Thus, a number is computable if there is a Turing machine that generates a sequence of rational approximations to the number.

A number is measurable if there is a Turing machine connected to the experiment that also produces rational approximations to that number — for the bisection method, the sequence of queries is that sequence of rational approximations.

The relation between the measurable and the non-measurable is as subtle as the relation between the computable and the non-computable. From what is non-measurable we can produce measurable numbers by suitable encoding. The same with the non-computable. Geroch and Hartle stresses this fact by giving the interesting example of a computable number made of non-computable numbers (see [15]):

$$M = \sum_{n=1}^{\infty} \frac{3^{-n}}{s(n)} , \tag{21}$$

where $s(n)$ is the number of steps taken by the Turing machine encoded in n to halt. This function s is itself non-computable. However, the number M is computable. In order to approximate the number M to within error, say $\varepsilon = 0.01$, it suffices to deal only with the first ten terms in the sum, and, even for these, only either to determine $s(n)$ or else ensure that it exceeds $1,000$. So, given $\varepsilon = 0.01$, our machine merely runs the first ten Turing machines for $1,000$ steps each one, letting $s(n)$ be infinite for any machine that has not by then halted.

8 Conclusions

This paper is about measurement seen from a computational point of view. In our models of Turing machines with physical oracles, introduced in our papers [2, 1, 5, 3, 4], we have been observing that our experiments make measurements (e.g., in [1] and [6]).

In [12, 16, 13], we find an established theory of measurement, axiomatized by Hempel in [16] extended by Carnap in [13]. Campbell, in [12], discusses the problem of measurement in experiments involving objects with almost identical attribute values.

According to a our framework all depends upon the physical theory chosen. For Newtonian mechanics we have shown that for some experimental quantities are *always* measurable (see [9, 5]) whilst for others there are quantities that are *not* always measurable. Our technical results can be used to show that the task of measuring quantities in physics can be classified by well known complexity classes. Principle 6, and the postulates, lead to a deeper understanding of experimenters and experiments which impose a *theoretical* and absolute limit on the measurability of a physical quantity.

In this paper we solved two problems: we were able to strongly root the ideas and results developed in [1, 4] in the Philosophy of Physics; and we were able to provide a decidable theory by adding time complexity measures into the Hempel's system of axioms.

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